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Boundary interpolation and rigidity for generalized Nevanlinna functions

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Dedicated to the memory of Erhard Schmidt

We solve a boundary interpolation problem at a real point for generalized Nevanlinna functions, and use the result to prove uniqueness theorems for generalized Nevanlinna functions.

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1 Introduction

In [18] D. Burns and S. Krantz proved the following *rigidity* (or uniqueness) result: Let s be a function holomorphic and contractive in the open unit disk \mathbb{D} such that

$$s(z) = z + O((1 - z)^4)$$

when z tends to 1 in an unrestricted way in \mathbb{D} . Then

$$s(z) \equiv z.$$

This result was generalized to multipoint conditions by V. Bolotnikov [11]. It was extended in [8] for the single point case to generalized Schur functions, that is, to functions s meromorphic in the open unit disk and such that the kernel

$$K_s(z, w) = \frac{1 - s(z)s(w)^*}{1 - zw^*}, \quad z, w \in \text{hol}(s),$$

has a finite number of negative squares; $\text{hol}(s)$ denotes the domain of holomorphy of s . Recall that a complex-valued function $K(z, w)$ defined on a set Ω is said to have $\kappa < \infty$ negative squares if it is Hermitian:

$$K(z, w) = K(w, z)^*, \quad z, w \in \Omega,$$

and if for every choice of an integer m and of points $z_1, \dots, z_m \in \Omega$, the $m \times m$ Hermitian matrix with ij -entry equal to $K(z_i, z_j)$ has at most κ negative eigenvalues, and exactly κ negative eigenvalues for some choice of m

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and z_1, \dots, z_m . We recall (see, for example, [5]) that associated to a function $K(z, w)$ with κ negative squares there is a unique reproducing kernel Pontryagin space with negative index κ and reproducing kernel $K(z, w)$, which we denote by $\mathcal{P}(K)$. This result originates from the paper of L. Schwartz [30]; see also [31].

The purpose of the present paper is to consider the counterpart of the results of [8] when one replaces the open unit disk by the open upper half-plane \mathbb{C}^+ . Recall that a function which is holomorphic on \mathbb{C}^+ and with values in $\mathbb{C}^+ \cup \mathbb{R}$ is called a Nevanlinna function. Equivalently, a Nevanlinna function n is characterized by the positivity of the kernel

$$L_n(z, w) = \frac{n(z) - n(w)^*}{z - w^*}, \quad z, w \in \mathbb{C}^+. \quad (1.1)$$

We denote by \mathbf{N}_0 the set of Nevanlinna functions. Recall that, by the Riesz-Herglotz representation theorem (see [26]), $n \in \mathbf{N}_0$ if and only if it can be written as

$$n(z) = \alpha + \beta z + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\sigma(t), \quad (1.2)$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$, and σ is a nondecreasing function such that

$$\int_{\mathbb{R}} \frac{d\sigma(t)}{1 + t^2} < \infty.$$

Formula (1.2) implies that setting

$$n(z^*) = n(z)^*,$$

we obtain a holomorphic extension of n to $\mathbb{C} \setminus \mathbb{R}$ such that

$$L_n(z, w) = \beta + \int_{\mathbb{R}} \frac{d\sigma(t)}{(t - z)(t - w^*)}, \quad z, w \in \mathbb{C} \setminus \mathbb{R}.$$

Thus the kernel L_n stays positive there.

A function n meromorphic in \mathbb{C}^+ is called a generalized Nevanlinna function if the kernel L_n in (1.1) has a finite number of negative squares in its domain of holomorphy in \mathbb{C}^+ . For instance, the function $n(z) = 1/z$ is a generalized Nevanlinna function and the associated kernel

$$L_n(z, w) = -\frac{1}{zw^*}, \quad z, w \in \mathbb{C}^+,$$

has one negative square. We always extend the generalized Nevanlinna function n to the lower open half-plane by $n(z^*) = n(z)^*$, $z \in \mathbb{C}^+$, and to those real points to which n can be continued by holomorphy. The extended function, still denoted by n , is locally meromorphic in $\mathbb{C} \setminus \mathbb{R}$ and for it the kernel L_n on $\text{hol}(n)$ in \mathbb{C} , defined by

$$L_n(z, w) = \begin{cases} \frac{n(z) - n(w)^*}{n - w^*}, & z \neq w^*, \\ n'(z), & z = w^*, \end{cases}$$

has the same number of negative squares as the original kernel L_n on \mathbb{C}^+ . The set of all generalized Nevanlinna functions n such that L_n has κ negative squares is denoted by \mathbf{N}_κ , and we put

$$\mathbf{N} = \bigcup_{\kappa \in \mathbb{N} \cup \{0\}} \mathbf{N}_\kappa.$$

These classes and their operator-valued generalizations, as well as related classes of functions, were introduced by M. G. Krein and H. Langer [27, 28]. For $n \in \mathbf{N}$ we define $\kappa_-(n) = \kappa$ if $n \in \mathbf{N}_\kappa$; if $\kappa_-(n) = 0$, the function n is a Nevanlinna function.

If a function $n \in \mathbf{N}$ is rational it is real on the real axis, and the total multiplicity of its poles (including the pole at ∞) is the *degree* of n , denoted by $\deg n$. The reproducing kernel Pontryagin space which is associated

with $n \in \mathbb{N}$ is denoted by $\mathcal{L}(n)$; its elements are functions meromorphic in $\mathbb{C} \setminus \mathbb{R}$. If $n \in \mathbb{N}$ is rational then $\dim \mathcal{L}(n) = \deg n$.

In this paper we consider the following boundary interpolation problems at a finite real point z_1 :

For given integer $k \geq 1$ and real numbers $\nu_0, \nu_1, \dots, \nu_{2k-1}$, find all generalized Nevanlinna functions n such that

$$n(z) = \nu_0 + \nu_1(z - z_1) + \dots + \nu_{2k-1}(z - z_1)^{2k-1} + R(z), \quad z \xrightarrow{\widehat{}} z_1, \quad (1.3)$$

with remainder term

$$R(z) = o((z - z_1)^{2k-1}) \quad \text{or} \quad O((z - z_1)^{2k}).$$

Using methods from the theory of reproducing kernel Pontryagin spaces, we give a description of all solutions of these problems in the case that the Hankel matrix H_k (see (3.2) below) associated with the coefficients in (1.3) is invertible. For the problem (1.3) restricted to Nevanlinna functions n we prove an existence and uniqueness result without invertibility of H_k ; see Theorem 8.1. This theorem also follows from a more general theorem in [23, Theorem 1]; while D. R. Georgijevic uses the theory of reproducing kernel Hilbert spaces, for our special case we use facts about Hankel matrices. Finally, applying the interpolation results for generalized Nevanlinna functions and the uniqueness statement for Nevanlinna functions, we prove a rigidity theorem for generalized Nevanlinna functions. In a forthcoming paper [3] boundary interpolation at a real point will be considered by means of extension theory of symmetric operators.

An outline of the paper is as follows. A useful fact on convergence in Pontryagin spaces is reviewed in Section 2. In Section 3 we formulate the boundary interpolation problems (see Problems 3.1 and 3.2) and review some related facts about Nevanlinna functions. Some technical lemmas about non-tangential limits of a generalized Nevanlinna function n and its kernel L_n , and their derivatives, are formulated in Section 4. The solutions of Problems 3.1 and 3.2 with invertible H_k are described in Section 5; proofs are given in Section 6. A special and important case of these problems, called the *basic interpolation problem*, is considered in Section 7. In Section 8 we prove the existence and uniqueness theorem for Nevanlinna functions mentioned above. Finally, the rigidity theorems are presented in Section 9.

After our paper was completed, we became aware of a recent paper by V. Bolotnikov [12], where rigidity statements for generalized Schur functions and generalized Nevanlinna functions are proved in the multi-point case. However, the methods in [12] are quite different from ours.

2 Convergence in Pontryagin spaces

In this section we recall a characterization of convergence in Pontryagin spaces from I. Iohvidov, M. Krein and H. Langer [25] and adapt it to reproducing kernel spaces. A *Pontryagin space* \mathcal{P} is a vector space \mathcal{P} endowed with a nondegenerate Hermitian form $[\cdot, \cdot]$ which can be decomposed into a direct and orthogonal sum

$$\mathcal{P} = \mathcal{P}_+ [+] \mathcal{P}_-, \quad (2.1)$$

such that the vector spaces \mathcal{P}_\pm equipped with the inner product $\pm[\cdot, \cdot]$ are Hilbert spaces and one of them, in this paper always \mathcal{P}_- , is finite dimensional. The space \mathcal{P} endowed with the inner product

$$\langle f, f \rangle := [f_+, f_+] - [f_-, f_-], \quad (2.2)$$

where $f = f_+ + f_-$ is the decomposition of f along (2.1), is a Hilbert space. If \mathcal{P} is not a Hilbert space itself, the decomposition (2.1) is not unique, but the dimension of \mathcal{P}_- , called the *negative index* of \mathcal{P} and denoted by $\kappa_-(\mathcal{P})$, is independent of the decomposition (2.1), and the topologies defined by the various decompositions are all equivalent; see for instance [9, Theorem 7.19]. In [25, Theorem 2.4] a characterization of convergent sequences in a Pontryagin space is given, using only the indefinite inner product. We will need this result in the following form; we sketch the proof.

Lemma 2.1 *Let (f_j) be a sequence of elements in a Pontryagin space \mathcal{P} such that $\lim_{j \rightarrow \infty} [f_j, f_j]$ exists and*

$$\lim_{j \rightarrow \infty} [f_j, g] \quad (2.3)$$

exists for all g in a dense subspace \mathcal{G} of \mathcal{P} . Then the limit in (2.3) exists for all $g \in \mathcal{P}$, and the sequence (f_j) converges weakly to some $f \in \mathcal{P}$. If moreover

$$\lim_{j \rightarrow \infty} [f_j, f_j] = [f, f], \quad (2.4)$$

then f_j converges strongly to f .

Proof. Since \mathcal{G} is dense, it contains a maximal negative subspace of \mathcal{P} of dimension $\kappa_-(\mathcal{P})$; see [25, Lemma 2.1]. We denote this subspace by \mathcal{G}_- , and consider the decomposition

$$\mathcal{P} = \mathcal{G}_-^{[\perp]} \oplus \mathcal{G}_-;$$

let $f_j = f_{j,+} + f_{j,-}$ be the corresponding decomposition of the element $f_j \in \mathcal{P}$. The existence of the limit in (2.3) for all $g \in \mathcal{G}_-$ and the fact that \mathcal{G}_- is finite dimensional imply that the strong limit $\lim_{j \rightarrow \infty} f_{j,-} =: f_-$ exists. Then also the limit $\lim_{j \rightarrow \infty} [f_{j,-}, f_{j,-}]$ exists, and the relation $[f_j, f_j] = [f_{j,+}, f_{j,+}] + [f_{j,-}, f_{j,-}]$ implies that also $\lim_{j \rightarrow \infty} [f_{j,+}, f_{j,+}]$ exists. By (2.3), the limit

$$\lim_{j \rightarrow \infty} [f_{j,+}, g] \quad (2.5)$$

exists for $g \in \mathcal{G} \cap \mathcal{G}_-^{[\perp]}$. Since $\mathcal{G} \cap \mathcal{G}_-^{[\perp]}$ is a dense subspace of the positive space $\mathcal{G}_-^{[\perp]}$ and the sequence $([f_{j,+}, f_{j,+}])$ is bounded, the limit in (2.5) exists for all $g \in \mathcal{G}_-^{[\perp]}$, that is, the sequence $(f_{j,+})$ converges weakly to an element $f_+ \in \mathcal{G}_-^{[\perp]}$, say. Hence the sequence (f_j) converges weakly to $f := f_+ + f_-$. Now assume that (2.4) holds. Then, in terms of the positive definite inner product (2.2), we have

$$\begin{aligned} \langle f - f_j, f - f_j \rangle &= [f - f_j, f - f_j] + 2[f_- - f_{j,-}, f_- - f_{j,-}] \\ &= [f, f] - [f, f_j] - [f_j, f] + [f_j, f_j] + 2[f_- - f_{j,-}, f_- - f_{j,-}] \longrightarrow 0, \quad j \longrightarrow \infty, \end{aligned}$$

which shows that the sequence (f_j) converges strongly to f . \square

Remark 2.2 If the space \mathcal{P} in Lemma 2.1 is a reproducing kernel Pontryagin space of functions on a set Ω with reproducing kernel K , the condition (2.3) can be replaced by the condition that $\lim_{j \rightarrow \infty} f_j(w)$ exists for all $w \in \Omega$. This follows from the relation

$$f_j(w) = [f_j, K(\cdot, w)], \quad w \in \Omega,$$

and the fact that the set of functions $z \mapsto K(z, w)$, $w \in \Omega$, is total in \mathcal{P} .

We shall also use the following result; see [2, Lemma 2.1].

Lemma 2.3 *Let $K(z, w)$ be a Hermitian complex-valued function holomorphic in z and w^* in some domain $\Omega \subset \mathbb{C}$ and with a finite number of negative squares. Then for $w \in \Omega$ the function*

$$z \mapsto \frac{\partial^j}{\partial w^{*j}} K(z, w), \quad z \in \Omega,$$

belongs to the associated reproducing kernel Pontryagin space $\mathcal{P}(K)$, and for every $f \in \mathcal{P}(K)$ we have

$$\left[f, \frac{\partial^j}{\partial w^{*j}} K(\cdot, w) \right] = f^{(j)}(w), \quad w \in \Omega.$$

3 Boundary interpolation for generalized Nevanlinna functions at a finite point

In the sequel, $\mathbb{N} = \{1, 2, \dots\}$ and $z \widehat{\rightarrow} z_1$ denotes the nontangential limit at z_1 , which means that z stays in a punctured sector of the form

$$S_\alpha = \left\{ z \in \mathbb{C} : z = z_1 + \rho e^{i\theta}, \theta \in \left[\frac{\pi}{2} - \alpha, \frac{\pi}{2} + \alpha \right], \rho > 0 \right\}, \quad \alpha \in [0, \pi/2). \quad (3.1)$$

For a finite or infinite sequence of real numbers ν_1, ν_2, \dots and $j \in \mathbb{N}$ we define the Hankel matrix

$$H_j = \begin{pmatrix} \nu_1 & \nu_2 & \cdots & \nu_{j-1} & \nu_j \\ \nu_2 & \nu_3 & \cdots & \nu_j & \nu_{j+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \nu_j & \nu_{j+1} & \cdots & \nu_{2j-2} & \nu_{2j-1} \end{pmatrix} \in \mathbb{R}^{j \times j}. \quad (3.2)$$

We consider the following interpolation problem.

Problem 3.1 Given $k \in \mathbb{N}$ and real numbers $z_1, \nu_0, \nu_1, \dots, \nu_{2k-1}$ such that the Hankel matrix H_k in (3.2) is invertible. Find all generalized Nevanlinna functions n such that

$$n(z) = \nu_0 + \nu_1(z - z_1) + \cdots + \nu_{2k-1}(z - z_1)^{2k-1} + o((z - z_1)^{2k-1}), \quad z \widehat{\rightarrow} z_1. \quad (3.3)$$

As a particular case, we will also be interested in the set of generalized Nevanlinna functions n which solve the following interpolation problem.

Problem 3.2 Given $k \in \mathbb{N}$ and real numbers $z_1, \nu_0, \nu_1, \dots, \nu_{2k-1}$ such that the Hankel matrix H_k in (3.2) is invertible. Find all generalized Nevanlinna functions n such that

$$n(z) = \nu_0 + \nu_1(z - z_1) + \cdots + \nu_{2k-1}(z - z_1)^{2k-1} + O((z - z_1)^{2k}), \quad z \widehat{\rightarrow} z_1. \quad (3.4)$$

In Theorems 5.3 and 5.4 below the solutions of these two problems are described in terms of a common linear fractional transformation, with an appropriate set of parameters in each case. The proofs of these theorems will be given in Section 6. For generalized Schur functions a condition of the type $O((z - z_1)^{2k})$ was considered in [4] and [8]; the results in these papers can easily be adapted to the case of a condition of the form $o((z - z_1)^{2k-1})$ as in Problem 3.1.

Clearly if n satisfies (3.4), then it also satisfies (3.3). That these conditions are not equivalent can be seen from an example where n is even a Nevanlinna function, the example will be given at the end of this section. For a rational function n each of the conditions (3.3) and (3.4) implies that n is holomorphic at z_1 and hence these conditions are equivalent and the given numbers $\nu_0, \nu_1, \dots, \nu_{2k-1}$ are the first $2k$ Taylor coefficients of n .

In the next lemma we recall some known characterizations of the expansion (3.3) for Nevanlinna functions.

Lemma 3.3 For $n \in \mathbf{N}_0$ with integral representation (1.2) and $k \in \mathbb{N}$ the following statements are equivalent:

(i) There exist $2k$ real numbers $\nu_0, \nu_1, \dots, \nu_{2k-2}, \nu_{2k-1}$ such that

$$n(z) = \nu_0 + \nu_1(z - z_1) + \cdots + \nu_{2k-2}(z - z_1)^{2k-2} + \nu_{2k-1}(z - z_1)^{2k-1} + o((z - z_1)^{2k-1}), \quad z \widehat{\rightarrow} z_1.$$

(ii) There exist $2k - 1$ real numbers $\nu_0, \nu_1, \dots, \nu_{2k-2}$ such that

$$n(z) = \nu_0 + \nu_1(z - z_1) + \cdots + \nu_{2k-2}(z - z_1)^{2k-2} + O((z - z_1)^{2k-1}), \quad z \widehat{\rightarrow} z_1.$$

(iii) $\int_{\mathbb{R}} \frac{d\sigma(t)}{(t - z_1)^{2k}} < \infty$.

If n has these properties then

$$\nu_0 = n(z_1) = \alpha + \beta z_1 + \int_{\mathbb{R}} \left(\frac{1}{t - z_1} - \frac{t}{1 + t^2} \right) d\sigma(t), \quad (3.5)$$

$$\nu_1 = \beta + \int_{\mathbb{R}} \frac{d\sigma(t)}{(t - z_1)^2}, \quad \nu_j = \int_{\mathbb{R}} \frac{d\sigma(t)}{(t - z_1)^{j+1}}, \quad j = 2, 3, \dots, 2k - 1. \quad (3.6)$$

Proof. (iii) \Rightarrow (i): The assumption (iii) implies that the numbers ν_j in (3.5) and (3.6) are all well defined and real. Now write $1/(t-z)$ as

$$\frac{1}{t-z} = \frac{1}{t-z_1} + \frac{z-z_1}{(t-z)^2} + \cdots + \frac{(z-z_1)^r}{(t-z_1)^{r+1}} + \frac{1}{t-z} \frac{(z-z_1)^{r+1}}{(t-z_1)^{r+1}}, \quad r = 0, 1, \dots, \quad (3.7)$$

and integrate, then with $r = 2k-1$

$$n(z) = \nu_0 + \nu_1(z-z_1) + \cdots + \nu_{2k-1}(z-z_1)^{2k-1} + f(z),$$

where

$$f(z) = (z-z_1)^{2k-1} \int_{\mathbb{R}} \frac{z-z_1}{t-z} \frac{d\sigma(t)}{(t-z_1)^{2k}} = o((z-z_1)^{2k-1}), \quad z \xrightarrow{\widehat{}} z_1.$$

The last equality holds by the bounded convergence theorem and the inequality

$$\left| \frac{z-z_1}{t-z} \right| = \left| \frac{z-z_1}{\operatorname{Im} z} \right| \left| \frac{\operatorname{Im} z}{t-z} \right| \leq \left| \frac{z-z_1}{\operatorname{Im} z} \right| \leq \frac{1}{\cos \alpha}, \quad t \in \mathbb{R}, \quad z \in S_\alpha. \quad (3.8)$$

Clearly, the asymptotic relation in (i) uniquely determines the coefficients ν_j , $j = 1, \dots, 2k-1$.

The implication (i) \Rightarrow (ii) is evident.

(ii) \Rightarrow (iii): Assume there exist real numbers $\mu_0, \mu_1, \dots, \mu_{2k-2}$ such that

$$n(z) = \mu_0 + \mu_1(z-z_1) + \cdots + \mu_{2k-2}(z-z_1)^{2k-2} + O((z-z_1)^{2k-1}), \quad z \xrightarrow{\widehat{}} z_1.$$

Then

$$n(z) - \sum_{i=0}^{2j} \mu_i (z-z_1)^i = O((z-z_1)^{2j+1}), \quad z \xrightarrow{\widehat{}} z_1, \quad j = 0, 1, \dots, k-1.$$

We claim that for $j = 0, \dots, k-1$

$$(a)_j \quad n(z) - \sum_{i=0}^{2j} \mu_i (z-z_1)^i = O((z-z_1)^{2j+1}), \quad z \xrightarrow{\widehat{}} z_1 \quad \Rightarrow \quad \begin{cases} \int_{\mathbb{R}} \frac{d\sigma(t)}{(t-z_1)^{2j+2}} < \infty & (b)_j, \\ \mu_i = \nu_i, \quad i = 0, \dots, 2j & (c)_j, \end{cases}$$

where the numbers ν_i are defined by (3.5) and (3.6). To prove the claim we use induction.

Consider the relation $(a)_0$. Setting $z = z_1 + iy$, $y > 0$, taking imaginary parts, and using that μ_0 is real, we obtain

$$\beta + \int_{\mathbb{R}} \frac{d\sigma(t)}{(t-z_1)^2 + y^2} = O(1), \quad y \downarrow 0.$$

The monotone convergence theorem now implies $(b)_0$. Hence the numbers ν_0 and ν_1 defined by (3.5) and (3.6) are finite and the implication (iii) \Rightarrow (i) implies

$$n(z) = \nu_0 + \nu_1(z-z_1) + o((z-z_1)), \quad z \xrightarrow{\widehat{}} z_1.$$

From this and the relation $(a)_0$ it follows that $\mu_0 = \nu_0$, that is, $(c)_0$ holds.

Now assume that for $j < k-1$ the implication $(a)_j \Rightarrow (b)_j, (c)_j$ holds.

Consider the relation $(a)_{j+1}$. Then $(b)_j$ and $(c)_j$ hold and the implication (iii) \Rightarrow (i) implies

$$n(z) = \nu_0 + \nu_1(z-z_1) + \cdots + \nu_{2j+1}(z-z_1)^{2j+1} + o((z-z_1)^{2j+1}), \quad z \xrightarrow{\widehat{}} z_1,$$

with ν_{2j+1} given by (3.6). Comparing this relation with the relation $(a)_{j+1}$ we find $\mu_i = \nu_i$ also for $i = 2j+1$.

From $(a)_{j+1}$ and (3.7) with $r = 2j+1$ we obtain

$$\int_{\mathbb{R}} \frac{(z-z_1)^{2j+2}}{t-z} \frac{d\sigma(t)}{(t-z_1)^{2j+2}} - \mu_{2j+2}(z-z_1)^{2j+2} = O((z-z_1)^{2j+3}), \quad z \xrightarrow{\widehat{}} z_1.$$

If we set $z = z_1 + iy$, $y > 0$, and consider imaginary parts, we obtain

$$\int_{\mathbb{R}} \frac{1}{(t - z_1)^2 + y^2} \frac{d\sigma(t)}{(t - z_1)^{2j+2}} = O(1), \quad y \downarrow 0.$$

Again by the monotone convergence theorem we get $(b)_{j+1}$. It follows that the numbers ν_i from (3.5) and (3.6) are finite for $i = 0, \dots, 2j + 3$ and the implication (iii) \Rightarrow (i) implies

$$n(z) = \nu_0 + \nu_1(z - z_1) + \dots + \nu_{2j+3}(z - z_1)^{2j+3} + o((z - z_1)^{2j+3}), \quad z \xrightarrow{\widehat{}} z_1.$$

Comparing this relation with the relation $(a)_{j+1}$ we obtain $(c)_{j+1}$. This completes the proof of the claim, hence (ii) implies $(b)_{k-1}$, that is, (iii). \square

Remark 3.4 Assume $n \in \mathbf{N}_0$ with integral representation (1.2) satisfies the asymptotic relation (3.3).

(1) It follows from the formula for ν_1 in (3.6) that $\nu_1 \geq 0$ and that if $\nu_1 = 0$ then n is identically equal to a constant ($= \alpha = \nu_0$) and also $\nu_j = 0$, $j = 2, \dots, 2k - 1$, and hence all the Hankel matrices H_j are zero. Therefore, when Nevanlinna functions are considered, the case that $\nu_1 = 0$ can often be excluded.

(2) The relations (3.6) imply that the $k \times k$ Hankel matrix H_k from (3.2) can be written in the form

$$H_k = \mathbf{e}_1 \mathbf{e}_1^* \beta + \int_{\mathbb{R}} \mathbf{p}(t) \mathbf{p}(t)^* d\sigma(t), \quad (3.9)$$

where

$$\mathbf{e}_1 = (1 \quad 0 \quad \dots \quad 0)^{\top} \in \mathbb{C}^k, \quad \mathbf{p}(t) = \left(\frac{1}{t - z_1} \quad \frac{1}{(t - z_1)^2} \quad \dots \quad \frac{1}{(t - z_1)^k} \right)^{\top},$$

and hence $H_k \geq 0$. If H_k is singular, we claim that n is rational with $\deg n \leq k - 1$. By Lemma 8.3 we have in fact $\deg n = \text{rank } H_k$. To prove the claim, consider a nonzero vector $\mathbf{x} \in \ker H_k$. Then (3.9) implies

$$0 = \mathbf{x}^* H_k \mathbf{x} = |\mathbf{x}^* \mathbf{e}_1|^2 \beta + \int_{\mathbb{R}} \frac{|q(t)|^2}{(t - z_1)^{2k}} d\sigma(t)$$

with the polynomial $q(t) = (t - z_1)^k \mathbf{x}^* \mathbf{p}(t)$. If the first component of \mathbf{x} is not zero then $\beta = 0$, q is a polynomial of degree $k - 1$ and hence σ can increase only in the at most $k - 1$ zeros of q , which implies the claim; if the first component of \mathbf{x} is zero then q is a polynomial of degree $\leq k - 2$, and since β can be $\neq 0$ we have again $\deg n \leq k - 1$.

(3) According to (3.6), the numbers ν_j for $j \geq 1$ can be considered as moments of the measure σ with respect to the functions $\frac{1}{(t - z_1)^{j+1}}$ (compare with [29, III.1]). Therefore the results in Section 5 have analogies to results for the finite Hamburger moment problem.

Example 3.5 Let σ be a bounded function with support in $[0, a]$, $a > 0$, such that

$$\int_0^a \frac{d\sigma(t)}{t^2} < \infty, \quad \int_0^a \frac{d\sigma(t)}{t^3} = \infty. \quad (3.10)$$

Then the Nevanlinna function

$$n(z) = \int_0^a \frac{d\sigma(t)}{t - z}$$

satisfies (3.3) but does not satisfy (3.4) with $z_1 = 0$ and $k = 1$. To see this we note that, by Lemma 3.3,

$$\nu_0 = n(0) = \int_0^a \frac{d\sigma(t)}{t}, \quad \nu_1 = \int_0^a \frac{d\sigma(t)}{t^2},$$

and hence

$$n(z) - \nu_0 - \nu_1 z = z^2 \int_0^a \frac{d\sigma(t)}{t^2(t - z)}.$$

By the bounded convergence theorem and the inequality (3.8) it readily follows that the function on the left-hand side is $o(z)$ as $z \widehat{\rightarrow} 0$. We claim that this function is not $O(z^2)$ as $z \widehat{\rightarrow} 0$. Assume the claim is false and that there is a number $K > 0$ such that for sufficiently small z in some sector S_α

$$\left| \frac{n(z) - \nu_0 - \nu_1 z}{z^2} \right| = \left| \int_0^a \frac{d\sigma(t)}{t^2(t-z)} \right| \leq K.$$

Then, setting $z = iy$, $y > 0$, and taking the real part of the integrand, we obtain that for every $\varepsilon > 0$

$$\int_\varepsilon^a \frac{d\sigma(t)}{t(t^2 + y^2)} \leq K.$$

If we let y tend to 0, then we see that for all $\varepsilon > 0$

$$\int_\varepsilon^a \frac{d\sigma(t)}{t^3} \leq K,$$

which contradicts the last assumption on σ in (3.10).

4 Non-tangential limits of functions and kernels

The following three lemmas will be used in the sequel. They are similar to results from [13, Section 7]. For a Nevanlinna function they readily follow from its integral representation (1.2) and the formulas (3.5) and (3.6). We consider a more general case.

Lemma 4.1 *For a function f which is holomorphic in $\mathcal{U}(z_1) \cap \mathbb{C}^+$, where $\mathcal{U}(z_1)$ is a neighborhood of z_1 in \mathbb{C} , the following statements are equivalent.*

- (i) $f(z) = \nu_0 + \nu_1(z - z_1) + \cdots + \nu_{2k-1}(z - z_1)^{2k-1} + o((z - z_1)^{2k-1})$, $z \widehat{\rightarrow} z_1$.
- (ii) $\lim_{z \widehat{\rightarrow} z_1} \frac{1}{i!} \frac{d^i}{dz^i} f(z) = \nu_i$, $i = 0, 1, \dots, 2k-1$.
- (iii) For some (and then for all) $j \in \{1, \dots, 2k-1\}$

$$\lim_{z \widehat{\rightarrow} z_1} \frac{1}{i!} \frac{d^i}{dz^i} \left(\frac{f(z) - \nu_0 - \nu_1(z - z_1) - \cdots - \nu_{j-1}(z - z_1)^{j-1}}{(z - z_1)^j} \right) = \nu_{i+j}, \quad i = 0, \dots, 2k-1-j.$$

The equivalence (i) \Leftrightarrow (ii) is shown in [32, Sections III.8 and III.9], the equivalence of these statements with (iii) then readily follows. For similar statements, see [14, Lemma 6].

The following lemma is a counterpart of [4, Lemma 2.1].

Lemma 4.2 *If the function $n \in \mathbf{N}$ admits the asymptotic expansion (3.3), then*

$$L_n(z, w) = \sum_{\substack{i,j=0 \\ i+j \leq 2k-2}}^{2k-2} \nu_{i+j-1} (z - z_1)^i (w^* - z_1)^j + o(\max\{|z - z_1|^{2k-2}, |w - z_1|^{2k-2}\}), \quad z, w \widehat{\rightarrow} z_1; \quad (4.1)$$

if the assumption (3.3) is replaced by (3.4) then also the last summand in (4.1) has to be replaced by

$$O(\max\{|z - z_1|^{2k-2}, |w - z_1|^{2k-2}\}).$$

Proof. We have

$$\begin{aligned} L_n(z, w) &= \sum_{i=0}^{2k-1} \nu_i \frac{(z - z_1)^i - (w^* - z_1)^i}{z - w^*} + \frac{o((z - z_1)^{2k-1}) + o((w^* - z_1)^{2k-1})}{z - w^*} \\ &= \sum_{i=1}^{2k-1} \nu_i \left(\sum_{j=0}^{i-1} (z - z_1)^j (w^* - z_1)^{i-1-j} \right) + \frac{o((z - z_1)^{2k-1}) + o((w^* - z_1)^{2k-1})}{z - w^*} = \end{aligned}$$

$$= \sum_{\substack{i,j=0 \\ i+j \leq 2k-2}}^{2k-2} \nu_{i+j+1} (z - z_1)^i (w^* - z_1)^j + \frac{o((z - z_1)^{2k-1}) + o((w^* - z_1)^{2k-1})}{z - w^*}.$$

Consider the last term in this expression. With S_α as in (3.1) and

$$z = z_1 + u, \quad w = z_1 + v, \quad z, w \in S_\alpha,$$

for $u, v \rightarrow 0$ the quotients $\frac{u}{u-v^*}$, $\frac{v^*}{u-v^*}$ remain bounded, and we obtain

$$\begin{aligned} \frac{o((z - z_1)^{2k-1}) + o((w^* - z_1)^{2k-1})}{z - w^*} &= \frac{o(u^{2k-1}) + o((v^*)^{2k-1})}{u - v^*} \\ &= \frac{u o(u^{2k-2}) + v^* o((v^*)^{2k-2})}{u - v^*} \\ &= o(u^{2k-2}) + o(v^{2k-2}) \\ &= o(\max\{u^{2k-2}, v^{2k-2}\}). \end{aligned}$$

□

Using Lemma 4.2 we obtain the following result.

Lemma 4.3 *If the function $n \in \mathbb{N}$ admits the asymptotic expansion (3.3), then*

$$\lim_{z \xrightarrow{z_1} z} \frac{\partial^{i+j}}{\partial z^i \partial w^{*j}} L_n(z, w) \Big|_{w=z} = \nu_{i+j+1}, \quad i, j = 0, \dots, k-1.$$

Proof. With S_α , defined by (3.1), consider a sequence $(w_p) \subset S_\alpha$ with $\lim_{p \rightarrow \infty} w_p = z_1$. Set $r_p = |w_p - z_1|$, choose real numbers $\beta \in (\alpha, \pi/2)$ and $\gamma \in (0, \sin(\beta - \alpha))$, and denote by Γ_p the circle

$$\Gamma_p = \{\zeta \in \mathbb{C} : |\zeta - w_p| = \gamma r_p\}.$$

Then $\Gamma_p \subset S_\beta$, because the distance of w_p to the boundary of S_β equals $r_p \sin \tau$ for some $\tau \in [\beta - \alpha, \beta]$ and hence $r_p \sin \tau \geq r_p \sin(\beta - \alpha) > r_p \gamma$.

We set

$$\nu_{i+j+1}(p) = \left(\frac{1}{2\pi i}\right)^2 \iint_{\Gamma_p \times \Gamma_p} \frac{\sum_{\substack{r,s=0 \\ r+s \leq 2k-2}}^{2k-2} \nu_{r+s+1} (\zeta_1 - z_1)^r (\zeta_2^* - z_1)^s}{(\zeta_1 - w_p)^{i+1} (\zeta_2^* - w_p)^{j+1}} d\zeta_1 d\zeta_2^*.$$

Using Cauchy's formula for functions of several complex variables (see for instance [19, (3.3)]) we can write

$$\begin{aligned} \nu_{i+j+1}(p) &= \frac{1}{i!j!} \frac{\partial^{i+j}}{\partial \zeta_1^i \partial \zeta_2^{*j}} \left(\sum_{\substack{r,s=0 \\ r+s \leq 2k-2}}^{2k-2} \nu_{r+s+1} (\zeta_1 - z_1)^r (\zeta_2^* - z_1)^s \right) \Big|_{\zeta_1 = \zeta_2 = w_p} \\ &= \nu_{i+j+1} + o(1), \quad p \rightarrow \infty, \end{aligned}$$

and obtain

$$\lim_{p \rightarrow \infty} \nu_{i+j+1}(p) = \nu_{i+j+1}.$$

Further, again using Cauchy's formula,

$$\begin{aligned} \frac{\partial^{i+j}}{\partial z^i \partial w^{*j}} L_n(z, w) \Big|_{z=w=w_p} &- \nu_{i+j+1}(p) \\ &= \left(\frac{1}{2\pi i}\right)^2 \iint_{\Gamma_p \times \Gamma_p} \frac{N(\zeta_1, \zeta_2)}{(\zeta_1 - w_p)^{i+1} (\zeta_2^* - w_p)^{j+1}} d\zeta_1 d\zeta_2^* \end{aligned}$$

with numerator

$$N(\zeta_1, \zeta_2) = L_n(\zeta_1, \zeta_2) - \sum_{\substack{r,s=0 \\ r+s \leq 2k-2}}^{2k-2} \nu_{r+s+1} (\zeta_1 - z_1)^r (\zeta_2^* - z_1)^s.$$

By Lemma 4.2, we have for $\zeta_1, \zeta_2 \in \Gamma_p$

$$N(\zeta_1, \zeta_2) = o(r_p^{2k-2}), \quad p \rightarrow \infty,$$

thus the above double integral is bounded in absolute value by a term of the form

$$o(r_p^{2k-2}) r_p^2 \frac{1}{(\gamma r_p)^{i+j+2}} = o(r_p^{2k-2-i-j}), \quad p \rightarrow \infty,$$

which tends to 0 as $p \rightarrow \infty$ since $i + j \leq 2k - 2$. □

5 Solution of the Problems 3.1 and 3.2

As in Problems 3.1 and 3.2, in this section we always assume that the matrix H_k from (3.2) is invertible. Following [4], to solve these problems we use reproducing kernel Pontryagin spaces of the kind introduced for the positive definite case by L. de Branges and J. Rovnyak in [15, 16]. In this method, there are three main steps. Here and throughout in the sequel

$$J_\ell = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (5.1)$$

- (1) Construct from the interpolation data a 2×2 $(-iJ_\ell)$ -unitary rational matrix function Θ and build a finite dimensional backward shift invariant Pontryagin space of rational functions with reproducing kernel of the form

$$\frac{J_\ell - \Theta(z)J_\ell\Theta(w)^*}{z - w^*}.$$

- (2) Show that the linear fractional transformation based on Θ gives a set of solutions for appropriate choices of the parameter.
- (3) Check that this set consists of all the solutions.

We define the matrix

$$C_k = \begin{pmatrix} \nu_0 & \nu_1 & \cdots & \nu_{k-1} \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{2 \times k} \quad (5.2)$$

and the shift matrix

$$Z_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{k \times k}. \quad (5.3)$$

Then, with the matrix J_ℓ from (5.1), the Lyapunov equation

$$H_k Z_k - Z_k^* H_k = -C_k^* J_\ell C_k \quad (5.4)$$

holds, which can be easily verified. In the following, the $\mathbb{C}^{2 \times 2}$ -valued function

$$\Theta(z) = I_2 - C_k((z - z_1)I_k - Z_k)^{-1}H_k^{-1}C_k^*J_\ell \quad (5.5)$$

will play an essential role. Since $Z_k^k = 0$, we have

$$((z - z_1)I_k - Z_k)^{-1} = \frac{1}{z - z_1} \left(I - \frac{Z_k}{z - z_1} \right)^{-1} = \sum_{j=0}^{k-1} \frac{Z_k^j}{(z - z_1)^{j+1}},$$

and it follows that $\Theta(z)$ is a rational function which has a pole at most at z_1 . Evidently,

$$\Theta(\infty) = I_2. \quad (5.6)$$

Moreover, $\Theta(z)$ satisfies the relation

$$\frac{J_\ell - \Theta(z)J_\ell\Theta(w)^*}{z - w^*} = C_k((z - z_1)I_k - Z_k)^{-1}H_k^{-1}((w - z_1)I_k - Z_k)^{-*}C_k^*. \quad (5.7)$$

Indeed,

$$\begin{aligned} \Theta(z)J_\ell\Theta(w)^* &= \left(I_2 - C_k((z - z_1)I_k - Z_k)^{-1}H_kJ_\ell \right) J_\ell \\ &\quad \times \left(I_2 + J_\ell C_k H_k^{-1}((w^* - z_1)I_k - Z_k^*)^{-1}C_k^* \right) \\ &= J_\ell + C_k((z - z_1)I_k - Z_k)^{-1}H_k^{-1}C_k^* \\ &\quad - C_k H_k^{-1}((w^* - z_1)I_k - Z_k^*)^{-1}C_k^* \\ &\quad + C_k((z - z_1)I_k - Z_k)^{-1}H_k^{-1}C_k^* J_\ell C_k H_k^{-1}((w^* - z_1)I_k - Z_k^*)^{-1}C_k^* \\ &= J_\ell C_k((z - z_1)I_k - Z_k)^{-1}H_k^{-1} \\ &\quad \times \{ ((w^* - z_1)I_k - Z_k^*)H_k - H_k((z - z_1)I_k - Z_k) + C_k^* J_\ell C_k \} \\ &\quad \times H_k^{-1}((w^* - z_1)I_k - Z_k^*)^{-1}C_k^*. \end{aligned}$$

Using the Lyapunov equation (5.4), we see that the last expression equals

$$J_\ell + (w^* - z)C_k((z - z_1)I_k - Z_k)^{-1}H_k^{-1}((w^* - z_1)I_k - Z_k^*)^{-1}C_k^*,$$

and (5.7) is proved. It implies that Θ takes $(-iJ_\ell)$ -unitary values on the real line. From this and the fact that, as observed above, Θ is rational with a pole at most in the real point z_1 , its determinant $\det \Theta(z)$ is a constant of modulus one (see for example [2, Theorem 3.6]) and from (5.6) we conclude that $\det \Theta$ is normalized, that is,

$$\det \Theta(z) = 1, \quad z \in \mathbb{C} \setminus \{z_1\}. \quad (5.8)$$

The kernel appearing in (5.7) is denoted by K_Θ :

$$K_\Theta(z, w) = \frac{J_\ell - \Theta(z)J_\ell\Theta(w)^*}{z - w^*}, \quad z, w \in \mathbb{C} \setminus \{z_1\}, \quad z \neq w^*. \quad (5.9)$$

It follows from (5.7) that the linear span of the functions $K_\Theta(\cdot, w)\mathbf{c}$, $w \in \mathbb{C} \setminus \{z_1\}$ and $\mathbf{c} \in \mathbb{C}^2$, is finite dimensional; denote this span by $\mathcal{P}(\Theta)$.

Lemma 5.1 *The space $\mathcal{P}(\Theta)$ is k -dimensional and spanned by the linearly independent functions*

$$\begin{aligned} \mathbf{f}_0(z) &= \frac{\begin{pmatrix} \nu_0 \\ 1 \end{pmatrix}}{z - z_1}; \\ \mathbf{f}_j(z) &= \frac{\begin{pmatrix} \nu_0 \\ 1 \end{pmatrix}}{(z - z_1)^{j+1}} + \frac{\begin{pmatrix} \nu_1 \\ 0 \end{pmatrix}}{(z - z_1)^j} + \cdots + \frac{\begin{pmatrix} \nu_j \\ 0 \end{pmatrix}}{z - z_1}, \quad j = 1, \dots, k-1. \end{aligned} \quad (5.10)$$

The inner product of two of these functions is given by

$$[\mathbf{f}_i, \mathbf{f}_j] = \nu_{i+j+1}, \quad i, j = 0, 1, \dots, k-1. \quad (5.11)$$

Proof. Let \mathcal{M} be the space spanned by the functions $\mathbf{f}_j, j = 0, 1, \dots, k-1$, and equipped with the inner product (5.11). Since H_k is invertible, \mathcal{M} is a k -dimensional Pontryagin space with negative index $\kappa_-(H_k)$. Moreover, \mathcal{M} , being a finite-dimensional inner product space whose elements are functions, is a reproducing kernel space; in this case the kernel is given by

$$K(z, w) = F(z)H_k^{-1}F(w)^*,$$

where F is the matrix function with columns $\mathbf{f}_j, j = 0, 1, \dots, k-1$, that is,

$$F(z) = (\mathbf{f}_0(z) \quad \mathbf{f}_1(z) \quad \dots \quad \mathbf{f}_{k-1}(z)) = C_k((z - z_1)I_k - Z_k)^{-1} \quad (5.12)$$

with C_k and Z_k from (5.2) and (5.3). By (5.7), $K = K_\Theta$, the kernel from (5.9), hence $\mathcal{M} = \mathcal{P}(\Theta)$. \square

If we write Θ as

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}, \quad (5.13)$$

then the functions a, b, c and d are rational with at most a pole in z_1 . In fact, the relations (5.13) and (5.5) imply by a straightforward calculation

$$a(z) = 1 + \sum_{j=1}^k \sum_{i=0}^{k-j} \nu_i \frac{\gamma_{i+j,1}}{(z - z_1)^j}, \quad b(z) = - \sum_{j=1}^k \sum_{i=0}^{k-j} \nu_i \frac{m_{i+j}}{(z - z_1)^j}, \quad (5.14)$$

$$c(z) = \sum_{j=1}^k \frac{\gamma_{j1}}{(z - z_1)^j}, \quad d(z) = 1 - \sum_{j=1}^k \frac{m_j}{(z - z_1)^j}, \quad (5.15)$$

where the γ_{ij} 's are the elements of the inverse matrix H_k^{-1} :

$$H_k^{-1} = (\gamma_{\ell j})_{\ell, j=1, \dots, k}, \quad (5.16)$$

and

$$m_j = \sum_{p=1}^k \nu_{p-1} \gamma_{jp}, \quad j = 1, \dots, k. \quad (5.17)$$

With the orthogonal rational functions as introduced in [17, Chapter 11] these relations become the analogs of the Christoffel-Darboux formulas. This will be considered elsewhere.

From (5.16) and (5.17) it follows that the numbers γ_{j1} and m_j appearing in the formulas (5.14) and (5.15) satisfy the relations

$$H_k \begin{pmatrix} \gamma_{11} \\ \gamma_{21} \\ \vdots \\ \gamma_{k1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad H_k \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_k \end{pmatrix} = \begin{pmatrix} \nu_0 \\ \nu_1 \\ \vdots \\ \nu_{k-1} \end{pmatrix}. \quad (5.18)$$

Hence, in particular, γ_{k1} does not depend on ν_0 , but if $\gamma_{k1} \neq 0$, then, since

$$m_k = \nu_0 \gamma_{k1} + \nu_1 \gamma_{k2} + \dots + \nu_{k-1} \gamma_{kk}, \quad (5.19)$$

m_k depends on ν_0 . We need one more auxiliary result.

Lemma 5.2 Assume that the matrix H_k is invertible. Then the numbers γ_{k1} and m_k in (5.18) cannot be zero simultaneously: $|\gamma_{k1}| + |m_k| > 0$.

Proof. Observe that

$$\gamma_{k1} = (-1)^{k+1} \frac{\det H'_k}{\det H_k}, \quad (5.20)$$

where

$$H'_k = \begin{pmatrix} \nu_2 & \nu_3 & \cdots & \nu_{k-1} & \nu_k \\ \nu_3 & \nu_4 & \cdots & \nu_k & \nu_{k+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \nu_k & \nu_{k+1} & \cdots & \nu_{2k-3} & \nu_{2k-2} \end{pmatrix} \in \mathbb{R}^{(k-1) \times (k-1)}.$$

Hence $\gamma_{k1} = 0$ is equivalent to

$$\det H'_k = 0. \quad (5.21)$$

We use Sylvester's identity as given in [33, (7.0.2)]; see also [24, §2]: Let H be a $k \times k$ -matrix. If H is written as

$$H = \begin{pmatrix} \alpha & \mathbf{u}^T & \beta \\ \mathbf{v} & \widehat{H} & \mathbf{w} \\ \gamma & \mathbf{x}^T & \delta \end{pmatrix}$$

with $\widehat{H} \in \mathbb{C}^{(k-2) \times (k-2)}$, $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y} \in \mathbb{C}^{2 \times 1}$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, then

$$\begin{aligned} & (\det H)(\det \widehat{H}) \\ &= \det \begin{pmatrix} \alpha & \mathbf{u}^T \\ \mathbf{v} & \widehat{H} \end{pmatrix} \det \begin{pmatrix} \widehat{H} & \mathbf{w} \\ \mathbf{x}^T & \delta \end{pmatrix} - \det \begin{pmatrix} \mathbf{u}^T & \beta \\ \widehat{H} & \mathbf{w} \end{pmatrix} \det \begin{pmatrix} \mathbf{v} & \widehat{H} \\ \gamma & \mathbf{x}^T \end{pmatrix}. \end{aligned} \quad (5.22)$$

For

$$H = \begin{pmatrix} \nu_0 & \nu_1 & \cdots & \nu_{k-1} & \nu_k \\ \nu_1 & \nu_2 & \cdots & \nu_k & \nu_{k+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \nu_{k-1} & \nu_k & \cdots & \nu_{2k-2} & \nu_{2k-1} \\ \nu_k & \nu_{k+1} & \cdots & \nu_{2k-1} & \nu_{2k} \end{pmatrix}$$

we find

$$\begin{aligned} \widehat{H} &= H'_k, \quad \begin{pmatrix} \alpha & \mathbf{u}^T \\ \mathbf{v} & \widehat{H} \end{pmatrix} = \begin{pmatrix} \nu_0 & \nu_1 & \cdots & \nu_{k-1} \\ \nu_1 & \nu_2 & \cdots & \nu_k \\ \vdots & \vdots & & \vdots \\ \nu_{k-1} & \nu_k & \cdots & \nu_{2k-2} \end{pmatrix}, \\ \begin{pmatrix} \mathbf{u}^T & \beta \\ \widehat{H} & \mathbf{w} \end{pmatrix} &= \begin{pmatrix} \mathbf{v} & \widehat{H} \\ \gamma & \mathbf{x}^T \end{pmatrix} = H_k, \end{aligned}$$

and (5.22) implies

$$\begin{aligned} & (\det H)(\det H'_k) + (\det H_k)^2 \\ &= \det \begin{pmatrix} \nu_0 & \nu_1 & \cdots & \nu_{k-1} \\ \nu_1 & \nu_2 & \cdots & \nu_k \\ \vdots & \vdots & & \vdots \\ \nu_{k-1} & \nu_k & \cdots & \nu_{2k-2} \end{pmatrix} \det \begin{pmatrix} \nu_2 & \cdots & \nu_k & \nu_{k+1} \\ \vdots & & \vdots & \vdots \\ \nu_k & \cdots & \nu_{2k-2} & \nu_{2k-1} \\ \nu_{k+1} & \cdots & \nu_{2k-1} & \nu_{2k} \end{pmatrix}. \end{aligned} \quad (5.23)$$

Since

$$\begin{pmatrix} \nu_0 & \nu_1 & \cdots & \nu_{k-1} \\ \nu_1 & \nu_2 & \cdots & \nu_k \\ \vdots & \vdots & & \vdots \\ \nu_{k-1} & \nu_k & \cdots & \nu_{2k-2} \end{pmatrix} \begin{pmatrix} \gamma_{k1} \\ \gamma_{k2} \\ \vdots \\ \gamma_{kk} \end{pmatrix} = \begin{pmatrix} \nu_0 \gamma_{k1} + \nu_1 \gamma_{k2} + \cdots + \nu_{k-1} \gamma_{kk} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

under the assumption that $m_k = 0$ we have, by (5.19), that

$$\det \begin{pmatrix} \nu_0 & \nu_1 & \cdots & \nu_{k-1} \\ \nu_1 & \nu_2 & \cdots & \nu_k \\ \vdots & \vdots & & \vdots \\ \nu_{k-1} & \nu_k & \cdots & \nu_{2k-2} \end{pmatrix} = 0,$$

and if also $\gamma_{k1} = 0$, that is, if (5.21) holds, the relation (5.23) implies $\det H_k = 0$, a contradiction since H_k is supposed to be invertible. \square

In the next two theorems we describe the solutions n of the Problems 3.1 and 3.2 in terms of the fractional linear transformation

$$n(z) = \mathcal{T}_\Theta(\widehat{n})(z) = \frac{a(z)\widehat{n}(z) + b(z)}{c(z)\widehat{n}(z) + d(z)} \quad (5.24)$$

with a, b, c and d from (5.13); here we use the convention that

$$\mathcal{T}_\Theta(\infty)(z) = \frac{a(z)}{c(z)}.$$

Clearly, if we multiply the matrix function Θ by $(z - z_1)^k$ and denote

$$\widetilde{\Theta}(z) = (z - z_1)^k \Theta(z) = \begin{pmatrix} \widetilde{a}(z) & \widetilde{b}(z) \\ \widetilde{c}(z) & \widetilde{d}(z) \end{pmatrix}, \quad (5.25)$$

then instead of (5.24) we can also write

$$n(z) = \mathcal{T}_{\widetilde{\Theta}}(\widehat{n})(z) = \frac{\widetilde{a}(z)\widehat{n}(z) + \widetilde{b}(z)}{\widetilde{c}(z)\widehat{n}(z) + \widetilde{d}(z)}.$$

The coefficients $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$ are polynomials in z and

$$\widetilde{\Theta}(z_1) = \begin{pmatrix} \nu_0 \gamma_{k1} & -\nu_0 m_k \\ \gamma_{k1} & -m_k \end{pmatrix}.$$

Lemma 5.2 implies that $\widetilde{\Theta}(z_1) \neq 0$ and therefore the rational matrix function Θ has a pole of order k at z_1 . Recall that $\det \Theta$ is normalized (see (5.8)), however, $\det \widetilde{\Theta}$ is not, in fact, by (5.25),

$$\det \widetilde{\Theta}(z) = (z - z_1)^{2k}. \quad (5.26)$$

Nevertheless, in the calculations that follow it is sometimes more convenient to work with polynomials than with rational functions, and we shall also use $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$.

We mention that the choice of Θ as a rational function with the pole in z_1 corresponds to the fact that the elements of the solution matrix for the finite moment problem (which corresponds to the case $z_1 = \infty$) are polynomials in z and for the Nevanlinna-Pick problem with (nonreal) data points z_1, \dots, z_m they are rational functions with poles in $z_1^*, z_2^*, \dots, z_m^*$.

Recall that \mathbf{N} is the class of generalized Nevanlinna functions, and that for $n \in \mathbf{N}$ the number of negative squares of the kernel L_n is denoted by $\kappa_-(n)$. Clearly, if n is a real constant we have $\kappa_-(n) = 0$, and we set also $\kappa_-(\infty) = 0$. For an Hermitian matrix H , by $\kappa_-(H)$ we denote the number of its negative eigenvalues. The numbers γ_{k1} and m_k appear in the formulas (5.16)–(5.20). Lemma 5.2 implies that with the two Cases (a) or (b) in the following theorems all possibilities are covered.

Theorem 5.3 (a) Suppose that $\gamma_{k1} \neq 0$. Then the formula (5.24) gives a one-to-one correspondence between all solutions n of Problem 3.1 and all $\hat{n} \in \mathbf{N} \cup \{\infty\}$ such that

$$\lim_{z \rightarrow z_1} \frac{z - z_1}{\hat{n}(z) + \frac{d(z)}{c(z)}} = 0. \quad (5.27)$$

(b) Suppose that $m_k \neq 0$. Then the formula (5.24) gives a one-to-one correspondence between all solutions n of Problem 3.1 and all $\hat{n} \in \mathbf{N} \cup \{\infty\}$ such that

$$\lim_{z \rightarrow z_1} \frac{z - z_1}{\frac{1}{\hat{n}(z)} + \frac{c(z)}{d(z)}} = 0. \quad (5.28)$$

In both Cases (a) and (b), if $n = \mathcal{T}_\Theta(\hat{n})$ then

$$\kappa_-(n) = \kappa_-(H_k) + \kappa_-(\hat{n}).$$

We now turn to the solution of Problem 3.2.

Theorem 5.4 (a) Suppose that $\gamma_{k1} \neq 0$. Then the formula (5.24) gives a one-to-one correspondence between all solutions n of Problem 3.2 and all $\hat{n} \in \mathbf{N} \cup \{\infty\}$ such that

$$\liminf_{z \rightarrow z_1} \left| \hat{n}(z) - \frac{m_k}{\gamma_{k1}} \right| > 0. \quad (5.29)$$

(b) Suppose that $m_k \neq 0$. Then the formula (5.24) gives a one-to-one correspondence between all solutions n of Problem 3.2 and all $\hat{n} \in \mathbf{N} \cup \{\infty\}$ such that

$$\liminf_{z \rightarrow z_1} \left| \frac{1}{\hat{n}(z)} - \frac{\gamma_{k1}}{m_k} \right| > 0. \quad (5.30)$$

In both Cases (a) and (b), if $n = \mathcal{T}_\Theta(\hat{n})$ then

$$\kappa_-(n) = \kappa_-(H_k) + \kappa_-(\hat{n}).$$

In these theorems we use the convention that $1/\infty = 0$ and $1/0 = \infty$. Thus, in both theorems, in Case (a) $n = \mathcal{T}_\Theta(\infty) = a/c$ is a solution and in Case (b) $n = \mathcal{T}_\Theta(0) = b/d$ is a solution.

Remark 5.5 In both theorems, if \hat{n} is a rational function then also n is a rational function and

$$\deg n = k + \deg \hat{n}. \quad (5.31)$$

In this case \liminf in (5.29) and in (5.30) can be replaced by \lim . In this case also, if $\gamma_{k1} \neq 0$, then the relations (5.27) and (5.29) are equivalent, and, if $m_k \neq 0$, then the relations (5.28) and (5.30) are equivalent. This can be shown directly, but also follows from the fact that for rational functions n the asymptotic expansions (3.3) and (3.4) coincide.

Remark 5.6 If the conditions $\gamma_{k1} \neq 0$ and $m_k \neq 0$ are both met in either Theorem 5.3 or Theorem 5.4, then the solution set corresponding to part (a) coincides with the solution set corresponding to part (b). This follows from the one-to-one correspondence in the theorems, direct proofs are given in Section 6.

6 Proofs of the theorems and remarks in Section 5

In this section we give the proofs of Theorems 5.3 and 5.4 and Remarks 5.5 and 5.6.

Proof of Theorem 5.3. The proof is divided into seven steps.

Step 1: *In Case (a) the function $n_\infty = \mathcal{T}_\Theta(\infty) = a/c$ is a solution of Problem 3.1.*

Proof. Note that $n_\infty = \tilde{a}/\tilde{c}$, where, according to (5.13)–(5.15) and (5.25), \tilde{a} and \tilde{c} are the polynomials

$$\tilde{a}(z) = (z - z_1)^k + \sum_{j=1}^k \sum_{i=0}^{k-j} \nu_i \gamma_{i+j,1} (z - z_1)^{k-j}, \quad \tilde{c}(z) = \sum_{j=1}^k \gamma_{j1} (z - z_1)^{k-j},$$

and, by assumption, $\tilde{c}(z_1) = \gamma_{k1} \neq 0$. Evidently, n_∞ , being real and rational, belongs to \mathbb{N} (in fact, by Step 7 below, $n_\infty \in \mathbb{N}_\kappa$). To prove Step 1 it remains to show that

$$\frac{\tilde{a}(z)}{\tilde{c}(z)} = \sum_{p=0}^{2k-1} \nu_p (z - z_1)^p + o((z - z_1)^{2k-1}), \quad z \xrightarrow{\widehat{}} z_1,$$

or, equivalently, since we are dealing with polynomials,

$$\tilde{c}(z) \cdot \left(\sum_{p=0}^{2k-1} \nu_p (z - z_1)^p \right) = \tilde{a}(z) + O((z - z_1)^{2k}), \quad z \xrightarrow{\widehat{}} z_1. \quad (6.1)$$

We start with the left-hand side

$$\tilde{c}(z) \cdot \left(\sum_{p=0}^{2k-1} \nu_p (z - z_1)^p \right) = \sum_{(j,p) \in R} \nu_p \gamma_{j1} (z - z_1)^{k+p-j},$$

where $R = \{(j, p) : 1 \leq j \leq k, 0 \leq p \leq 2k - 1\}$. We divide R into three parts via the lines $p = j + 1/2$ and $p = j + k - 1/2$:

$$\begin{aligned} R_1 &= \{(j, p) : 1 \leq j \leq k, 0 \leq p \leq j - 1\} = \{(s + t, t) : 1 \leq s \leq k, 0 \leq t \leq k - s\}, \\ R_2 &= \{(j, p) : 1 \leq j \leq k, 0 \leq p - j \leq k - 1\} = \{(s, s + t) : 1 \leq s \leq k, 0 \leq t \leq k - 1\}, \\ R_3 &= \{(j, p) : 1 \leq j \leq k - 1, j + k \leq p \leq 2k - 1\}. \end{aligned}$$

Then the following equalities hold for $z \xrightarrow{\widehat{}} z_1$:

$$\sum_{(j,p) \in R_1} \nu_p \gamma_{j1} (z - z_1)^{k+p-j} = \sum_{s=1}^k \sum_{t=0}^{k-s} \nu_t \gamma_{s+t,1} (z - z_1)^{k-s} = \tilde{a}(z) - (z - z_1)^k,$$

$$\sum_{(j,p) \in R_2} \nu_p \gamma_{j1} (z - z_1)^{k+p-j} = \sum_{t=0}^{k-1} \sum_{s=1}^k \nu_{s+t} \gamma_{s1} (z - z_1)^{k-s} = (z - z_1)^k,$$

and

$$\sum_{(j,p) \in R_3} \nu_p \gamma_{j1} (z - z_1)^{k+p-j} = \sum_{j=1}^{k-1} \sum_{p=j+k}^{2k-1} \nu_p \gamma_{j1} (z - z_1)^{k+p-j} = O((z - z_1)^{2k}).$$

For the last equality in the middle we have applied the first relation in (5.18). Adding the three sums and using that $R = R_1 \cup R_2 \cup R_3$ we see that

$$\sum_{(j,p) \in R} \nu_p \gamma_{j1} (z - z_1)^{k+p-j} = \tilde{a}(z) + O((z - z_1)^{2k}), \quad z \xrightarrow{\widehat{}} z_1,$$

which is the right-hand side of (6.1). □

Step 2. In Case (a), a function n of the form (5.24) is a solution of Problem 3.1 if and only if (5.27) holds, that is, if and only if

$$\lim_{z \rightarrow z_1} \frac{z - z_1}{\widehat{n}(z) + \frac{d(z)}{c(z)}} = 0.$$

Proof. According to Step 1 $n = \mathcal{T}_\Theta(\infty) = a/c$ is a solution and the corresponding parameter $\widehat{n} = \infty$ satisfies this limit condition. Now let n be of the form (5.24) for some generalized Nevanlinna function $\widehat{n} \in \mathbf{N}$. Observing (5.25) and (5.26), we find

$$\begin{aligned} n(z) - \frac{a(z)}{c(z)} &= \frac{a(z)\widehat{n}(z) + b(z)}{c(z)\widehat{n}(z) + d(z)} - \frac{a(z)}{c(z)} = \frac{b(z)c(z) - a(z)d(z)}{c(z)(c(z)\widehat{n}(z) + d(z))} \\ &= -\frac{\det \widetilde{\Theta}(z)}{\widetilde{c}(z)^2 \left(\widehat{n}(z) + \frac{d(z)}{c(z)} \right)} = -\frac{(z - z_1)^{2k}}{\widetilde{c}(z)^2 \left(\widehat{n}(z) + \frac{d(z)}{c(z)} \right)}. \end{aligned} \quad (6.2)$$

Since a/c is a solution of Problem 3.1 and $\widetilde{c}(z_1) \neq 0$, we see that n is a solution of Problem 3.1 if and only if the parameter \widehat{n} satisfies

$$\frac{(z - z_1)^{2k}}{\widehat{n}(z) + \frac{d(z)}{c(z)}} = o((z - z_1)^{2k-1}), \quad z \rightarrow z_1,$$

that is, if and only if (5.27) is in force. □

Step 3. In Case (b) the function $n_0 = \mathcal{T}_\Theta(0) = b/d$ is a solution of Problem 3.1.

Proof. We have $n_0 = \widetilde{b}/\widetilde{d}$, where, according to (5.13)–(5.15) and (5.25), \widetilde{b} and \widetilde{d} are the polynomials

$$\widetilde{b}(z) = -\sum_{j=1}^k \sum_{i=0}^{k-j} \nu_i m_{i+j} (z - z_1)^{k-j}, \quad \widetilde{d}(z) = (z - z_1)^k - \sum_{j=1}^k m_j (z - z_1)^{k-j},$$

and, by assumption, $d(z_1) = -m_k \neq 0$. Since n_0 is rational and real, it belongs to \mathbf{N} , hence to prove Step 3 we only need to show that

$$\frac{\widetilde{b}(z)}{\widetilde{d}(z)} = \sum_{p=0}^{2k-1} \nu_\ell (z - z_1)^p + o((z - z_1)^{2k-1}), \quad z \rightarrow z_1,$$

or, equivalently, that

$$\widetilde{d}(z) \cdot \left(\sum_{p=0}^{2k-1} \nu_\ell (z - z_1)^p \right) = \widetilde{b}(z) + O((z - z_1)^{2k}), \quad z \rightarrow z_1.$$

We begin by the left-hand side and consecutively obtain for $z \rightarrow z_1$

$$\begin{aligned} \widetilde{d}(z) \cdot \left(\sum_{p=0}^{2k-1} \nu_\ell (z - z_1)^p \right) &= \sum_{p=0}^{2k-1} \nu_p (z - z_1)^{k+p} - \sum_{j=1}^k \sum_{p=0}^{2k-1} \nu_p m_j (z - z_1)^{k+p-j} \\ &= \sum_{p=0}^{k-1} \nu_p (z - z_1)^{k+p} - \sum_{j=1}^k \sum_{p=0}^{k+j-1} \nu_p m_j (z - z_1)^{k+p-j} + O((z - z_1)^{2k}) \\ &= \sum_{p=0}^{k-1} \nu_p (z - z_1)^{k+p} - \sum_{j=1}^k \left(\sum_{p=j}^{k+j-1} + \sum_{p=0}^{j-1} \right) \nu_p m_j (z - z_1)^{k+p-j} + O((z - z_1)^{2k}) = \end{aligned}$$

$$\begin{aligned}
&= \sum_{p=0}^{k-1} \nu_p (z - z_1)^{k+p} - \sum_{j=1}^k \sum_{p=0}^{k-1} \nu_{p+j} m_j (z - z_1)^{k+p} \\
&\quad - \sum_{s=1}^k \sum_{t=0}^{s-k} \nu_t m_{s+t} (z - z_1)^{k-s} + O((z - z_1)^{2k}) \\
&= \tilde{b}(z) + O((z - z_1)^{2k}).
\end{aligned}$$

The last equality holds because $\nu_p = \sum_{j=1}^k \nu_{p+j} m_j$, $p = 0, \dots, k-1$, according to the last relation in (5.18). \square

Step 4. In Case (b), a function n of the form (5.24) is a solution if and only if \hat{n} satisfies (5.28), that is, if and only if

$$\lim_{z \rightarrow z_1} \frac{z - z_1}{\frac{1}{\hat{n}(z)} + \frac{c(z)}{d(z)}} = 0.$$

Proof. By Step 3, $n = \mathcal{T}_\Theta(0) = b/d$ is a solution and the corresponding parameter $\hat{n} = 0$ satisfies the limit condition. Now let n be of the form (5.24) for some generalized Nevanlinna function $\hat{n} \in \mathbf{N} \setminus \{0\}$. Then using (5.25) and (5.26) we obtain

$$\begin{aligned}
n(z) - \frac{b(z)}{d(z)} &= \frac{a(z)\hat{n}(z) + b(z)}{c(z)\hat{n}(z) + d(z)} - \frac{b(z)}{d(z)} = \frac{(a(z)d(z) - b(z)c(z))\hat{n}(z)}{d(z)(c(z)\hat{n}(z) + d(z))} \\
&= \frac{\det \tilde{\Theta}(z)}{\tilde{d}(z)^2 \left(\frac{1}{\hat{n}(z)} + \frac{c(z)}{d(z)} \right)} = \frac{(z - z_1)^{2k}}{\tilde{d}(z)^2 \left(\frac{1}{\hat{n}(z)} + \frac{c(z)}{d(z)} \right)}. \tag{6.3}
\end{aligned}$$

Since d/c is a solution of Problem 3.1 and $\tilde{d}(z_1) \neq 0$, we see that n is a solution of Problem 3.1 if and only if the corresponding parameter \hat{n} satisfies

$$\frac{(z - z_1)^{2k}}{\frac{1}{\hat{n}(z)} + \frac{c(z)}{d(z)}} = o((z - z_1)^{2k-1}), \quad z \rightarrow z_1,$$

that is, if and only if (5.28) holds. \square

It remains to show that all solutions of Problem 3.1 are of the form $(a\hat{n} + b)/(c\hat{n} + d)$ with \hat{n} as described in the theorem. This is done in Steps 5 to 7 of this proof by making use of the corresponding reproducing kernel spaces.

Step 5. Assume that n is a solution of Problem 3.1. Then the functions

$$g_j(z) = \frac{n(z) - \sum_{i=0}^j \nu_i (z - z_1)^i}{(z - z_1)^{j+1}}, \quad j = 0, 1, \dots, k-1,$$

belong to $\mathcal{L}(n)$ and $[g_i, g_j] = \nu_{i+j+1}$, $i, j = 0, 1, \dots, k-1$.

Proof. Let (ϵ_p) be a sequence of positive numbers tending to 0, such that n is holomorphic in a neighborhood of every $w_p = z_1 + i\epsilon_p$, and define the functions

$$g_{j,p}(z) = \frac{1}{j!} \frac{\partial^j}{\partial w^{*j}} L_n(z, w) \Big|_{w=w_p}, \quad z \in \mathbb{C}^+, \quad j = 0, \dots, k-1.$$

By Lemma 2.3 they belong to $\mathcal{L}(n)$, and by Lemma 4.3

$$\lim_{p \rightarrow \infty} [g_{j,p}, g_{j,p}] = \frac{\partial^{2j}}{\partial z^j \partial w^{*j}} K(z, w) \Big|_{z=w=w_p} = \nu_{2j+1}.$$

Moreover,

$$\begin{aligned} g_{j,p}(z) &= \frac{n(z) - n(w_p)^*}{(z - w_p^*)^{j+1}} - \sum_{i=1}^j \frac{n^{(i)}(w_p)^* i!}{(z - w_p^*)^{j+1-i}} \\ &= \frac{n(z) - n(w_p)^* - \sum_{i=1}^j n^{(i)}(w_p)^* i! (z - w_p^*)^i}{(z - w_p^*)^{j+1}}, \end{aligned}$$

hence, since n is a solution of Problem 3.1,

$$\lim_{p \rightarrow \infty} g_{j,p}(z) = g_j(z), \quad z \in \mathbb{C}^+ \cap \text{hol}(n), \quad j = 0, 1, \dots, k-1. \quad (6.4)$$

Lemma 2.1 (see also Remark 2.2) implies that the sequence $(g_{j,p})$ converges weakly to some function in $\mathcal{L}(n)$ as $p \rightarrow \infty$. It follows from (6.4) that this limit function coincides with g_j , thus, in particular, $g_j \in \mathcal{L}(n)$. On account of Lemma 4.1 we have

$$[g_j, g_j] = \lim_{p \rightarrow \infty} [g_j, g_{j,p}] = \lim_{p \rightarrow \infty} g_j^{(j)}(w_p) = \nu_{2j+1} = \lim_{p \rightarrow \infty} [g_{j,p}, g_{j,p}].$$

Hence, by the last part of Lemma 2.1, we have that $(g_{j,p})$ converges strongly to g_j in $\mathcal{L}(n)$, $j = 0, 1, \dots, k-1$. This and Lemma 4.1 imply that

$$[g_i, g_j] = \lim_{p \rightarrow \infty} [g_{i,p}, g_{j,p}] = \nu_{i+j+1},$$

and the proof of Step 5 is complete. \square

Step 6. Assume that n is a solution of Problem 3.1. Then the map

$$\tau : \mathbf{f} \mapsto \begin{pmatrix} 1 & -n \end{pmatrix} \mathbf{f}, \quad \mathbf{f} \in \mathcal{P}(\Theta), \quad (6.5)$$

is an isometry from $\mathcal{P}(\Theta)$ into $\mathcal{L}(n)$.

Proof. Indeed, by Lemma 5.1 concerning the characterization of $\mathcal{P}(\Theta)$ and Step 5,

$$\begin{pmatrix} 1 & -n \end{pmatrix} \mathbf{f}_j = g_j, \quad j = 0, \dots, k-1,$$

and

$$[\mathbf{f}_i, \mathbf{f}_j]_{\mathcal{M}} = \nu_{i+j+1} = [g_i, g_j]_{\mathcal{L}(n)}.$$

\square

Step 7. If n is a solution of Problem 3.1, then it is of the form (5.24).

Proof. With J_ℓ as in (5.1) the kernel L_n can be written as follows.

$$\begin{aligned} L_n(z, w) &= \frac{\begin{pmatrix} 1 & -n(z) \end{pmatrix} J_\ell \begin{pmatrix} 1 & -n(w) \end{pmatrix}^*}{z - w^*} \\ &= \frac{\begin{pmatrix} 1 & -n(z) \end{pmatrix} \{J_\ell - \Theta(z) J_\ell \Theta(w)^*\} \begin{pmatrix} 1 & -n(w) \end{pmatrix}^*}{z - w^*} \\ &\quad + \frac{\begin{pmatrix} 1 & -n(z) \end{pmatrix} \Theta(z) J_\ell \Theta(w)^* \begin{pmatrix} 1 & -n(w) \end{pmatrix}^*}{z - w^*}. \end{aligned}$$

Since the map τ from (6.5) is an isometry, the sum

$$\mathcal{L}(n) = \tau \tau^* \mathcal{L}(n) + (1 - \tau \tau^*) \mathcal{L}(n) \quad (6.6)$$

is direct and orthogonal. The space $\tau\tau^*\mathcal{L}(n) = \tau\mathcal{P}(\Theta)$ is the reproducing kernel Pontryagin space with reproducing kernel

$$\frac{(1 \quad -n(z)) \{J_\ell - \Theta(z)J_\ell\Theta(w)^*\} (1 \quad -n(w))^*}{z - w^*}$$

and is isomorphic to $\mathcal{P}(\Theta)$, hence it is k -dimensional and has negative index $\kappa_-(H_k)$. The space $(1 - \tau\tau^*)\mathcal{L}(n)$ is the reproducing kernel Pontryagin space with reproducing kernel

$$\frac{(1 \quad -n(z)) \Theta(z)J_\ell\Theta(w)^* (1 \quad -n(w))^*}{z - w^*},$$

therefore this kernel has $\kappa_-(n) - \kappa_-(H_k)$ negative squares.

Now we define functions u and v by the relation

$$(u(z) \quad -v(z)) = (1 \quad -n(z)) \Theta(z). \quad (6.7)$$

Assume first that $u(z) \not\equiv 0$. In view of the relation

$$\frac{(1 \quad -n(z)) \Theta(z)J_\ell\Theta(w)^* (1 \quad -n(w))^*}{z - w^*} = u(z) \frac{\frac{v(z)}{u(z)} - \frac{v(w)^*}{u(w)^*}}{z - w^*} u(w)^*,$$

it follows that the function

$$\hat{n}(z) = \frac{v(z)}{u(z)}$$

is a generalized Nevanlinna function with index $\kappa_-(\hat{n}) = \kappa_-(n) - \kappa_-(H_k)$. Furthermore, (6.7) implies that

$$n(z) = \frac{a(z)\hat{n}(z) + b(z)}{c(z)\hat{n}(z) + d(z)},$$

that is, we get the linear fractional transformation (5.24).

Assume now $u(z) \equiv 0$. Then $n = a/c$, and it corresponds to $\hat{n} = \infty$ in the above representation of n . \square

Proof of Theorem 5.4. The proof can be given along the same lines as the proof of Theorem 5.3. Only the conclusions of Step 2 and Step 4 have to be modified. The conditions given in the theorem follow again from (6.2) in Case (a) and from (6.3) in Case (b). We explain the modification for Case (a), Step 2; the modification for Case (b), Step 4 is similar and omitted. By Step 1 in Case (a) the function a/c is a solution of Problem 3.2, hence $n \in \mathbf{N}$ is a solution of this interpolation problem if and only if

$$n(z) - \frac{a(z)}{c(z)} = O((z - z_1)^{2k}), \quad z \widehat{\rightarrow} z_1.$$

Now assume that n is of the form (5.24) with parameter \hat{n} . Then, on account of (6.2), n is a solution of Problem 3.2 if and only if \hat{n} satisfies

$$\frac{1}{\tilde{c}(z)^2 (\hat{n}(z) + \frac{d(z)}{c(z)})} = O(1), \quad z \widehat{\rightarrow} z_1. \quad (6.8)$$

Since $\tilde{c}(z_1) = \gamma_{k1} \neq 0$ and

$$\lim_{z \widehat{\rightarrow} z_1} \frac{d(z)}{c(z)} = \lim_{z \widehat{\rightarrow} z_1} \frac{\tilde{d}(z)}{\tilde{c}(z)} = -\frac{m_k}{\gamma_{k1}} =: \rho$$

the relation (6.8) is equivalent to the relation

$$\frac{1}{\hat{n}(z) + \rho} = O(1), \quad z \widehat{\rightarrow} z_1,$$

and this in turn is equivalent to (5.29). \square

Proof of Remark 5.5. The equality (5.31) follows from the decomposition (6.6) in Step 7 of the proof of Theorem 5.3 and the discussion following it. For they imply that $\mathcal{L}(n)$ is isomorphic to the orthogonal sum $\mathcal{P}(\Theta) \oplus \mathcal{L}(\hat{n})$ if $\hat{n} = v/u$ and to $\mathcal{P}(\Theta)$ if $\hat{n} = \infty$, which corresponds to the case that $u = 0$. The proof of the last statement is left to the reader. \square

Proof of Remark 5.6. If in Theorem 5.3 or in Theorem 5.4 the conditions in (a) and (b) hold simultaneously, then $\rho = -m_k/\gamma_{k1} \in \mathbb{R} \setminus \{0\}$. As to Theorem 5.3 if both assumptions in (a) and (b) hold, then the equivalence of the limits (5.27) and (5.28) for $\hat{n} \in \mathbb{N} \setminus \{0\}$ follows from the equality

$$\frac{\beta}{\alpha + \beta} = 1 - \frac{1/\beta}{\frac{1}{\alpha} + \frac{1}{\beta}}, \quad \alpha, \beta \in \mathbb{C} \setminus \{0\}, \quad \alpha + \beta \neq 0, \quad (6.9)$$

with $\alpha = \hat{n}(z)$, $\beta = d(z)/c(z)$ and both sides multiplied by $(z - z_1)$, and the fact that

$$\lim_{z \rightarrow z_1} \frac{d(z)}{c(z)} = \rho.$$

For $\hat{n} \equiv 0$ and $\hat{n} = \infty$ both limit conditions (5.27) and (5.28) are satisfied.

Concerning Theorem 5.4 if both assumptions in (a) and (b) hold, then the equivalence between (5.29) and (5.30) for $\hat{n} \in \mathbb{N} \setminus \{0\}$ also follows from (6.9) but now with $\alpha = \hat{n}(z)$ and $\beta = \rho$; if $\hat{n} \equiv 0$ and $\hat{n} = \infty$ then (5.29) and (5.30) are satisfied. \square

We close this section with the following observation. If we replace ν_0 in the interpolation problem (3.1) by $\nu_0 + \alpha$, $\alpha \in \mathbb{R}$, then the linear fractional transformation (5.24) becomes

$$\begin{aligned} n(z) + \alpha &= \frac{(a(z) + \alpha c(z))\hat{n}(z) + b(z) + \alpha d(z)}{c(z)\hat{n}(z) + d(z)} \\ &= \frac{(a(z) + \alpha c(z))(\hat{n}(z) + \alpha) + (b(z) + \alpha d(z) - \alpha a(z) - \alpha a(z)c(z))}{c(z)(\hat{n}(z) + \alpha) + d(z) - \alpha c(z)}, \end{aligned}$$

corresponding to the new matrix function

$$\Theta_\alpha(z) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \Theta(z) \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix},$$

which also satisfies $\Theta_\alpha(\infty) = I_2$ and $\det \Theta_\alpha(z) = 1$, $z \in \mathbb{C} \setminus \{z_1\}$.

7 Basic interpolation problems

A particular case of Problem 3.1 is the following *basic interpolation problem*.

Problem 7.1 Given $k \in \mathbb{N}$ and $k + 2$ real numbers $z_1, \nu_0, \nu_1, \dots, \nu_{2k-1}$ with $\nu_k \neq 0$. Find all generalized Nevanlinna functions n such that

$$n(z) = \nu_0 + \nu_k(z - z_1)^k + \dots + \nu_{2k-1}(z - z_1)^{2k-1} + o((z - z_1)^{2k-1}), \quad z \xrightarrow{\widehat{}} z_1.$$

Problem 7.1 is indeed a special case of Problem 3.1 in that now

$$\nu_1 = \dots = \nu_{k-1} = 0, \quad \nu_k \neq 0, \quad (7.1)$$

which implies that H_k from (3.2) takes the form

$$H_k = \begin{pmatrix} 0 & 0 & \cdots & 0 & \nu_k \\ 0 & 0 & \cdots & \nu_k & \nu_{k+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \nu_k & \nu_{k+1} & \cdots & \nu_{2k-2} & \nu_{2k-1} \end{pmatrix} \in \mathbb{R}^{k \times k} \quad (7.2)$$

and is invertible. The term *basic* expresses the fact that under the assumption (7.1) the set $\nu_0, \nu_k, \dots, \nu_{2k-1}$ is a minimal set for which the above Problems 3.1, 3.2 can be solved; this corresponds to the fact that the $(-iJ_\ell)$ -unitary rational matrix function which defines the linear fractional transformation describing the set of all solutions of the basic Problem 7.1 (see (5.5) below) has no minimal $(-iJ_\ell)$ -unitary factorizations; see Theorem 7.4 below.

We first apply Theorem 5.3 to describe the solutions of Problem 7.1. Case (a) prevails, because, by (7.2) and the first relation in (5.18), $\gamma_{k1} = 1/\nu_k$, hence $\gamma_{k1} \neq 0$. Using the notation as in (5.16), namely that $H_k^{-1} = (\gamma_{\ell j})_{\ell,j=1,\dots,k}$, we define the real polynomial p by

$$p(z) = \gamma_{11}(z - z_1)^{k-1} + \gamma_{21}(z - z_1)^{k-2} + \dots + \gamma_{k1}. \quad (7.3)$$

Since the matrix C_k from (5.2) takes the form

$$C_k = \begin{pmatrix} \nu_0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix},$$

we find that Θ in (5.5) can be written as

$$\Theta(z) = I_2 + \frac{p(z)}{(z - z_1)^k} \begin{pmatrix} \nu_0 & \\ & 1 \end{pmatrix} J_\ell. \quad (7.4)$$

Hence the polynomials in

$$\tilde{\Theta}(z) = (z - z_1)^k \Theta(z) = \begin{pmatrix} \tilde{a}(z) & \tilde{b}(z) \\ \tilde{c}(z) & \tilde{d}(z) \end{pmatrix}$$

(see (5.25)) are given by

$$\begin{aligned} \tilde{a}(z) &= (z - z_1)^k + \nu_0 p(z), & \tilde{b}(z) &= -\nu_0^2 p(z), \\ \tilde{c}(z) &= p(z), & \tilde{d}(z) &= (z - z_1)^k - \nu_0 p(z). \end{aligned}$$

Thus, as a consequence of Theorem 5.3 (a) we obtain the following result.

Theorem 7.2 *The linear fractional transformation*

$$n(z) = \frac{((z - z_1)^k + \nu_0 p(z))\hat{n}(z) - \nu_0^2 p(z)}{p(z)\hat{n}(z) + (z - z_1)^k - \nu_0 p(z)} \quad (7.5)$$

with p from (7.3), gives a one-to-one correspondence between all solutions n of Problem 7.1 and all $\hat{n} \in \mathbf{N} \cup \{\infty\}$ such that

$$\lim_{z \xrightarrow{\sim} z_1} \frac{z - z_1}{\hat{n}(z) - \nu_0 + \frac{(z - z_1)^k}{p(z)}} = 0. \quad (7.6)$$

With n and \hat{n} as in (7.5) we have

$$\kappa_-(n) = \kappa_-(H_k) + \kappa_-(\hat{n}).$$

Remark 7.3 A basic interpolation problem corresponding to Problem 3.2 is the same as Problem 7.1 but with the term $o((z - z_1)^{2k-1})$ replaced by the term $O((z - z_1)^{2k})$. As to the solutions of this problem, Theorem 7.2 holds but with (7.6) replaced by

$$\liminf_{z \xrightarrow{\sim} z_1} |\hat{n}(z) - \nu_0| > 0.$$

This follows from Theorem 5.4 (a), the calculations preceding Theorem 7.2, and the equality $m_k/\gamma_{k1} = \nu_0$ obtained from (5.18) and (7.2).

We close this section with a theorem about elementary factors. By $\mathcal{U}_\ell^{z_1}$ we denote the class of all rational 2×2 matrix functions which are of the form

$$\Theta(z) = \sum_{j=0}^s \frac{T_j}{(z - z_1)^j}, \quad (7.7)$$

with an integer $s \geq 0$ and 2×2 matrices T_j , $j = 0, 1, \dots, s$, and which are $(-iJ_\ell)$ -unitary on $\mathbb{R} \setminus \{z_1\}$. If $\Theta \in \mathcal{U}_\ell^{z_1}$ has the form (7.7), then, by definition, its McMillan degree is given by

$$\deg \Theta = \text{rank} \begin{pmatrix} T_s & 0 & \cdots & 0 & 0 \\ T_{s-1} & T_s & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T_2 & T_3 & \cdots & T_s & 0 \\ T_1 & T_2 & \cdots & T_{s-1} & T_s \end{pmatrix};$$

see [10]. The class $\mathcal{U}_\ell^{z_1}$ is closed under multiplication. A product $\Theta_1 \Theta_2$ of two functions from this class is called *minimal* if $\deg \Theta_1 \Theta_2 = \deg \Theta_1 + \deg \Theta_2$. A product of functions from $\mathcal{U}_\ell^{z_1}$ is not automatically minimal, since $\mathcal{U}_\ell^{z_1}$ is closed under taking inverses. A function $\Theta \in \mathcal{U}_\ell^{z_1}$ is called *elementary* if in every minimal factorization $\Theta = \Theta_1 \Theta_2$ with factors $\Theta_1, \Theta_2 \in \mathcal{U}_\ell^{z_1}$, at least one of the factors is constant, hence of the form

$$e^{i\theta} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \theta, \alpha, \beta, \gamma, \delta \in \mathbb{R}, \quad \alpha\delta - \beta\gamma = 1.$$

Theorem 7.4 *The matrix function Θ from (7.4) belongs to $\mathcal{U}_\ell^{z_1}$ and is elementary.*

Proof. Since $\nu_0 = \dots = \nu_{k-1} = 0$, the space $\mathcal{P}(\Theta)$ is spanned by the functions

$$\mathbf{f}_j = \frac{\begin{pmatrix} \nu_0 \\ 1 \end{pmatrix}}{(z - z_1)^{j+1}}, \quad j = 0, \dots, k-1,$$

(see Lemma 5.1) and in terms of the matrix function F from (5.12) the inner product can be written as

$$[F\mathbf{c}, F\mathbf{d}] = \mathbf{d}^* H_k \mathbf{c}, \quad \mathbf{c}, \mathbf{d} \in \mathbb{C}^k,$$

where now H_k is given by (6.3). Recall that there is a one-to-one correspondence between the minimal $(-iJ_\ell)$ -unitary factorizations of Θ (up to multiplicative $(-iJ_\ell)$ -unitary factors) and the nondegenerate resolvent-invariant subspaces of $\mathcal{P}(\Theta)$; see [7, Theorem 6.2] and [6]. The space $\mathcal{P}(\Theta)$ is made of a single chain, and its resolvent-invariant subspaces are exactly the spaces

$$\mathcal{M}_j = \text{linear span } \{\mathbf{f}_0, \dots, \mathbf{f}_{j-1}\}, \quad j = 1, \dots, k.$$

The corresponding Gram matrix is given by the $j \times j$ principal submatrix H_j of H_k . In view of the form (7.2) of H_k , all principal submatrices H_j with $j = 1, \dots, k-1$ are singular, and thus there are no nontrivial minimal $(-iJ_\ell)$ -unitary factorizations. \square

It can be shown that any $\Theta \in \mathcal{U}_\ell^{z_1}$ can be written as a minimal product of elementary factors; see [2] and [20] for the case $z_1 = \infty$.

8 A boundary interpolation problem for Nevanlinna functions

In this section we consider a boundary interpolation problem as Problem 3.1 for Nevanlinna functions. Whereas the classical Nevanlinna–Pick interpolation problem (with given values at points in \mathbb{C}^+) is always solvable if a certain Hankel matrix containing the data is nonnegative, for the boundary interpolation problem (as for the finite moment problem) the condition $H_k \geq 0$ is in general not sufficient for the existence of a solution; see

Theorem 8.1. This theorem is a special case of [23, Theorem 1] which was proved there in the multi-point situation via reproducing kernel spaces. For the simpler case we present here we give a more direct proof. The uniqueness statement in Theorem 8.1 will be used to prove the rigidity results in the next section.

Recall that for a finite or infinite sequence of real numbers ν_1, ν_2, \dots we denote by H_j , $j = 1, 2, \dots$, the Hankel matrices H_j as in (3.2), that is,

$$H_j = \begin{pmatrix} \nu_1 & \nu_2 & \cdots & \nu_j \\ \nu_2 & \nu_3 & \cdots & \nu_{j+1} \\ \vdots & \vdots & & \vdots \\ \nu_j & \nu_{j+1} & \cdots & \nu_{2j-1} \end{pmatrix},$$

and we denote by H_∞ the infinite Hankel matrix $H_\infty = (\nu_{1+i+j})_{i,j \geq 0}$; sometimes we set $H_0 = 0$.

Theorem 8.1 Let $k \in \mathbb{N}$ and $\nu_0, \nu_1, \dots, \nu_{2k-1}$ be real numbers. There exists a Nevanlinna function n with asymptotic expansion

$$n(z) = \nu_0 + \nu_1(z - z_1) + \cdots + \nu_{2k-1}(z - z_1)^{2k-1} + o((z - z_1)^{2k-1}), \quad z \widehat{\rightarrow} z_1, \quad (8.1)$$

if and only if $H_k \geq 0$ and, if $\text{rank } H_k < k$ then $\text{rank } H_{k-1} = \text{rank } H_k$. In this case, if H_k is invertible there exist infinitely many Nevanlinna functions n with asymptotic expansion (8.1) (their description was given in Theorem 5.3), and if H_k is singular then n is unique and rational with degree equal to $\text{rank } H_k$.

Remark 8.2 A simple example showing that the condition $H_k \geq 0$ is not sufficient for the existence of a solution n is given by the matrix

$$H_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\nu_1 = 0$ and, according to Remark 3.4 (1), $n \equiv \nu_0$ is the only Nevanlinna function with $\nu_1 = 0$, which would imply $0 = \nu_3 \neq 1$.

In the proof of the theorem we make use of the following lemmas. The first one is well-known.

Lemma 8.3 Let f be a rational function which is holomorphic at z_1 and has Taylor expansion

$$f(z) = \sum_{i=0}^{\infty} \nu_i (z - z_1)^i.$$

The $\deg f = r$ can be characterized in the following equivalent ways:

- (i) r is the number of poles of f , counted according to their multiplicities and including ∞ if it is a pole.
 - (ii) r is the largest integer for which the Hankel matrix H_r corresponding to ν_1, ν_2, \dots is invertible; if no such matrix exists then $r = 0$.
 - (iii) r is the rank of the infinite Hankel matrix $H_\infty = (\nu_{1+i+j})_{i,j \geq 0}$.
- If, in addition, f is a Nevanlinna function with integral representation (1.2), then statements (i)–(iii) are equivalent to the statements

- (iv) $r = \begin{cases} \# \text{supp } \sigma & \text{if } \beta = 0, \\ \# \text{supp } \sigma + 1 & \text{if } \beta \neq 0. \end{cases}$
- (v) r is the number positive squares of the kernel

$$L_f(z, w) = \beta + \int_{\mathbb{R}} \frac{d\sigma(t)}{(t - z)(t - w^*)}.$$

The equivalence of statements (i), (ii), and (iii) follows, for example, from [22, Abschnitt 16.10, Satz 8, Folgerung], where the case $z_1 = \infty$ is considered. The equivalence between (iii) and (v) also follows from the representation

$$L_f(z, w) = \sum_{i,j=0}^{\infty} \nu_{1+i+j} (z - z_1)^i (w^* - z_1)^j.$$

Lemma 8.4 For $k \in \mathbb{N}$ and given real numbers $\nu_1, \nu_2, \dots, \nu_{2k-1}$, suppose that H_k (and hence also all the matrices H_1, H_2, \dots, H_{k-1}) are nonnegative. If $\nu_1 = 0$, then $\nu_1 = \nu_2 = \dots = \nu_{2k-2} = 0$, $\nu_{2k-1} \geq 0$. If for some $r \in \{1, \dots, k-1\}$ the matrix H_r is invertible (hence H_1, H_2, \dots, H_{r-1} are also invertible) and H_{r+1} is singular (hence $H_{r+2}, H_{r+3}, \dots, H_k$ are also singular), then

$$\text{rank } H_{r+1} = \text{rank } H_{r+2} = \dots = \text{rank } H_{k-1} = r, \quad \text{rank } H_k = r \text{ or } r+1.$$

Proof. We only consider the case where $\nu_1 > 0$. First we show that, for matrices $H_\ell, H_{\ell+1}, H_{\ell+2}$ from the sequence

$$H_1, H_2, \dots, H_k, \tag{8.2}$$

if H_ℓ is singular we have

$$\text{nul } H_{\ell+1} = \text{nul } H_\ell + 1,$$

where for example $\text{nul } H_\ell = \dim \ker H_\ell$. Let $\mathbf{x} \in \ker H_\ell$. Then $(\mathbf{x}^* \ 0) H_{\ell+1} \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} = 0$ and hence

$$H_{\ell+1} \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix} = 0. \tag{8.3}$$

Similarly it follows that also $H_{\ell+2} \begin{pmatrix} \mathbf{x} \\ 0 \\ 0 \end{pmatrix} = 0$, and this easily implies

$$H_{\ell+1} \begin{pmatrix} 0 \\ \mathbf{x} \end{pmatrix} = 0. \tag{8.4}$$

Thus, $\text{nul } H_{\ell+1} > \text{nul } H_\ell$. On the other hand the difference between these numbers can be at most one.

Now we consider the sequence (8.2). If $k = r+1$ there is nothing to prove, since $\text{nul } H_k = 1$. If $k = r+2$ then $\text{nul } H_{k-1} = 1$ and $\text{nul } H_k = 1$ or 2 . If $k = r+3$ we apply the above result for $\ell = r+1$, and it yields $\text{nul } H_{k-1} = \text{nul } H_{k-2} + 1$; to H_k it cannot be applied since no H_{k+1} is given. \square

Remark 8.5 Under the assumptions of Lemma 8.4, if $\text{rank } H_k = r+1$ then each $(k+1) \times (k+1)$ Hankel matrix H which contains H_k as principal submatrix has one negative eigenvalue.

The following statement is taken from [24, Theorem 9.2]; the Hankel matrices there are not necessarily non-negative.

Lemma 8.6 Let $\nu_1, \dots, \nu_{2\ell-1}$ be given real numbers such that H_ℓ is singular with rank r ($< \ell$) and if $r \geq 1$ then H_r is invertible. Then there exist unique real numbers $\nu_{2\ell}, \nu_{2\ell+1}$ such that the rank of the matrix $H_{\ell+1}$ equals r .

The last two lemmas imply the following corollary.

Corollary 8.7 If, for an infinite sequence ν_1, ν_2, \dots of real numbers, $H_\infty \geq 0$ and for some $r \in \mathbb{N}$ the matrix H_r is invertible and H_{r+1} is singular, then

$$r = \text{rank } H_r = \text{rank } H_{r+1} = \dots$$

and all matrices H_j , $j \geq r+2$, are uniquely determined by H_{r+1} .

Lemma 8.8 Suppose that for given real numbers $\nu_0, \nu_1 > 0, \nu_2, \dots, \nu_{2r+1}$ the matrix H_r is invertible and the matrix H_{r+1} is singular and nonnegative. Then there exists a rational function $n \in \mathbb{N}_0$ such that

$$n(z) = \nu_0 + \nu_1(z - z_1) + \dots + \nu_{2r+1}(z - z_1)^{2r+1} + o((z - z_1)^{2r+1}), \quad z \xrightarrow{\widehat{}} z_1. \tag{8.5}$$

Proof. We choose a sequence (ε_i) with $\varepsilon_i \downarrow 0$ when $i \rightarrow \infty$, and consider the numbers $\nu_1, \dots, \nu_{2r}, \nu_{2r+1} + \varepsilon_i$. For the corresponding Hankel matrix $H_{r+1}^{[i]}$ we have $H_{r+1}^{[i]} > 0$, hence according to Theorem 5.3, for each i there exist rational Nevanlinna functions $n^{[i]}$ having the property (8.5) with ν_{2r+1} replaced by $\nu_{2r+1} + \varepsilon_i$. Indeed, define the rational functions $a^{[i]}(z), b^{[i]}(z), c^{[i]}(z), d^{[i]}(z)$ by (5.14), (5.15) with k replaced by $r+1$ and coefficients $\gamma_j^{[i]}, m_j^{[i]}, j = 1, \dots, r+1$, from (5.18) with H_k replaced by $H_{r+1}^{[i]}$. If $\gamma_{r+1,1}^{[i]} \neq 0$ then the first and if $m_{r+1}^{[i]} \neq 0$ then the second of the functions

$$n^{[i]}(z) = \frac{a^{[i]}(z)}{c^{[i]}(z)}, \quad n^{[i]}(z) = \frac{b^{[i]}(z)}{d^{[i]}(z)},$$

has the asymptotic expansion (8.5) with ν_{2r+1} replaced by $\nu_{2r+1} + \varepsilon_i$. By Lemma 5.2 the numbers $\gamma_{r+1,1}^{[i]}$ and $m_{r+1}^{[i]}$ cannot be zero simultaneously. Furthermore, $\gamma_{r+1,1}^{[i]} \det H_{r+1}^{[i]}$ and $m_{r+1}^{[i]} \det H_{r+1}^{[i]}$ are independent of i , since they are determinants of matrices obtained from $H_{r+1}^{[i]}$ by replacing its last column by the right-hand sides of the equalities in (5.18). Therefore we can, for example, assume that

$$\gamma_{r+1} := \gamma_{r+1,1}^{[i]} \det H_{r+1}^{[i]} \neq 0$$

and hence that for all i the first function satisfies (8.5) with $\nu_{2r+1} + \varepsilon_i$ instead of ν_{2r+1} . If we write the function $n^{[i]}$ in the form

$$n^{[i]}(z) = \frac{a^{[i]}(z) \det H_r^{[i]}(z - z_1)^r}{c^{[i]}(z) \det H_r^{[i]}(z - z_1)^r}$$

then the numerator and denominator are polynomials whose coefficients (being the determinants of matrices obtained from $H_r^{[i]}$ by replacing one column by the right-hand side of the first relation in (5.18)) depend linearly on ε_i and hence have a limit when $i \rightarrow \infty$. The constant term in the denominator is γ_{r+1} . Since this number is nonzero, the rational functions $n^{[i]}$ converge in a neighborhood of z_1 to a rational function n with the given numbers $\nu_0, \nu_1, \dots, \nu_{2k-1}$ as its first Taylor coefficients. Clearly, as a limit of Nevanlinna functions n is also a Nevanlinna function. \square

Proof of Theorem 8.1. Necessity: Suppose n is a Nevanlinna function with the asymptotic expansion (8.1). Then by Remark 3.4 (2) the corresponding Hankel matrix H_k is nonnegative. It remains to consider the case $\text{rank } H_k < k$. Then, again by Remark 3.4 (2), the function n is rational, hence holomorphic at z_1 , and all matrices $H_j, j \geq 1$, corresponding to the Taylor coefficients of n at z_1 are nonnegative. We consider two cases:

$\nu_1 = 0$: Then by Remark 3.4 (1) $n \equiv \nu_0$ and $H_j = 0, j \geq 1$, hence, trivially, $\text{rank } H_{k-1} = \text{rank } H_k$.

$\nu_1 > 0$: Then there is an $r \in \{1, \dots, k-1\}$ such that H_r is invertible and H_{r+1} is singular. By Corollary 8.7, $\text{rank } H_{k-1} = \text{rank } H_k$.

Sufficiency: If H_k is invertible, according to Theorem 5.3 there are infinitely many functions $n \in \mathbb{N}_0$ obeying (8.1). From now on we assume H_k is not invertible. Again we consider two cases:

$\nu_1 = 0$: Then by the assumptions and Lemma 8.6, $H_k = 0$ and the function $n = \nu_0$ satisfies (8.1).

$\nu_1 > 0$: Then there is an $r \in \{1, \dots, k-1\}$ such that H_r is invertible and H_{r+1} is singular. According to Lemma 8.8, there exists a rational function n that satisfies (8.5), and by Lemma 8.3 $\deg n = r$. Since n is holomorphic at z_1 , to its Taylor coefficients there corresponds the infinite sequence of nonnegative Hankel matrices

$$H_1, H_2, \dots, H_r, H_{r+1}, H'_{r+2}, \dots, H'_k, \dots;$$

here we use the fact that the first $2r+2$ Taylor coefficients of n coincide with the given ν_0, \dots, ν_{2r+1} . According to Corollary 8.7,

$$r = \text{rank } H_{r+1} = \text{rank } H'_{r+2} = \dots$$

By assumption we have

$$r = \text{rank } H_{r+1} = \text{rank } H_{r+2} = \dots = \text{rank } H_k.$$

By Lemma 8.6, $H'_{r+2} = H_{r+2}, \dots, H'_k = H_k$, that is, n satisfies (8.1); by Corollary 8.7, it is unique. \square

Remark 8.9 If H_k , $k \geq 2$, is singular and r is the largest index ($< k-1$) such that H_r is invertible or $r = 0$ if no such index exists, then we have according to Lemma 8.4, $\text{rank } H_{r+1} = \text{rank } H_{r+2} = \dots = \text{rank } H_{k-1} = r$ and $\text{rank } H_k = r$ or $r+1$. If $\text{rank } H_k = r+1$ then there is no function $n \in \mathbf{N}_0$ with the asymptotic expansion (8.1), but there exists a function $\tilde{n} \in \mathbf{N}_0$ such that for $z \xrightarrow{\widehat{}} z_1$ we have

$$\tilde{n}(z) = \nu_0 + \nu_1(z - z_1) + \dots + \nu_{2k-2}(z - z_1)^{2k-2} + \tilde{\nu}_{2k-1}(z - z_1)^{2k-1} + o((z - z_1)^{2k-1})$$

with $\tilde{\nu}_{2k-1} < \nu_{2k-1}$. This corresponds to the fact that for the finite Hamburger moment problem, in the general solution for the last moment only an inequality may hold; see [29, Chapter V.1.1].

9 Rigidity for generalized Nevanlinna functions

In this section we give sufficient conditions under which a generalized Nevanlinna function is determined by the first terms of its asymptotic expansion at a real point z_1 . The simplest example of this is as follows: If n is a Nevanlinna function and for some $\nu_0 \in \mathbb{R}$,

$$n(z) = \nu_0 + o(z - z_1), \quad z \xrightarrow{\widehat{}} z_1,$$

then $n(z) \equiv \nu_0$; see Remark 3.4 (1) and, for the case $z_1 = \infty$, [1, Lemma 3.3.6]. We first formulate in Theorem 9.1 the rigidity statement for Nevanlinna functions. It is a reformulation of the uniqueness result in Theorem 8.1.

Theorem 9.1 *Let f be a rational Nevanlinna function of degree k which is holomorphic at z_1 . If n is a Nevanlinna function such that*

$$n(z) = f(z) + o((z - z_1)^{2k+1}), \quad z \xrightarrow{\widehat{}} z_1, \quad (9.1)$$

then $n = f$.

For example, if $n \in \mathbf{N}_0$ is such that

$$n(z) = -\frac{1}{z} + o((z - 1)^3), \quad z \xrightarrow{\widehat{}} 1,$$

then $n(z) = -1/z$.

Proof of Theorem 9.1. The assumptions and (9.1) imply that f and n have the same asymptotic expansion

$$\left. \begin{array}{l} n(z) \\ f(z) \end{array} \right\} = \nu_0 + \nu_1(z - z_1) + \dots + \nu_{2k-1}(z - z_1)^{2k-1} + \nu_{2k}(z - z_1)^{2k} + \nu_{2k+1}(z - z_1)^{2k+1} + o((z - z_1)^{2k+1}), \quad z \xrightarrow{\widehat{}} z_1,$$

with real coefficients $\nu_0, \nu_1, \dots, \nu_{2k+1}$. Let H_{k+1} be the corresponding Hankel matrix. Since f is rational of degree k we have $\det H_{k+1} = 0$; see Lemma 8.3. Theorem 8.1 with k replaced by $k+1$ now implies that $n = f$. \square

The following rigidity result for generalized Nevanlinna functions will be proved by reducing it to Theorem 9.1 and using the results of Section 5.

Theorem 9.2 *Let f be a rational generalized Nevanlinna function of degree $\ell \geq 1$. Assume f is holomorphic at z_1 with Taylor expansion*

$$f(z) = \sum_{j=0}^{\infty} \nu_j(z - z_1)^j,$$

and for some integer k with $1 \leq k \leq \ell$ the matrix H_k is invertible and $\kappa_-(f) = \kappa_-(H_k)$. If n is a generalized Nevanlinna function with $\kappa_-(n) = \kappa_-(f)$ such that

$$n(z) = f(z) + o((z - z_1)^{2\ell+1}), \quad z \xrightarrow{\widehat{}} z_1, \quad (9.2)$$

then $n = f$.

For example, if $n \in \mathbf{N}_1$ satisfies the relation

$$n(z) = \frac{1}{z} + o((z-1)^3), \quad z \widehat{\rightarrow} 1,$$

then $n(z) = 1/z$.

Proof of Theorem 9.2. The assumptions imply that n and f are solutions of Problem 3.2 with k as in the theorem. By Theorem 5.4,

$$n = \mathcal{T}_\Theta(\widehat{n}), \quad f = \mathcal{T}_\Theta(\widehat{f})$$

with parameters \widehat{n} , which is ∞ or a Nevanlinna function, and \widehat{f} , which is ∞ or a rational Nevanlinna function of degree $\ell - k$, satisfying (5.29) or (5.30).

To prove that $n = f$ we consider the two cases $\gamma_{k1} \neq 0$ and $\gamma_{k1} = 0$ separately.

Case (a): $\gamma_{k1} \neq 0$. Then

$$\rho = \lim_{z \widehat{\rightarrow} z_1} \frac{d(z)}{c(z)} = \lim_{z \widehat{\rightarrow} z_1} \frac{\widehat{d}(z)}{\widehat{c}(z)} = -\frac{\nu_0 \gamma_{k1} + \nu_1 \gamma_{k2} + \cdots + \nu_{k-1} \gamma_{kk}}{\gamma_{k1}} \in \mathbb{R}.$$

There are the following possibilities.

- 1) $\widehat{n} = \widehat{f} = \infty$. Then $n = f$.
- 2) $\widehat{n}, \widehat{f} \in \mathbf{N}_0$. Then, since these parameters satisfy (5.29),

$$\frac{1}{\widehat{n}(z) + \rho} = O(1), \quad \frac{1}{\widehat{f}(z) + \rho} = O(1), \quad z \widehat{\rightarrow} z_1. \quad (9.3)$$

The equalities (9.2), (5.25), and (5.26) imply that for $z \widehat{\rightarrow} z_1$

$$\begin{aligned} o((z - z_1)^{2k+1}) &= n(z) - f(z) = \frac{a(z)\widehat{n}(z) + b(z)}{c(z)\widehat{n}(z) + d(z)} - \frac{a(z)\widehat{f}(z) + b(z)}{c(z)\widehat{f}(z) + d(z)} \\ &= \frac{(a(z)d(z) - b(z)c(z))(\widehat{n}(z) - \widehat{f}(z))}{(c(z)\widehat{n}(z) + d(z))(c(z)\widehat{f}(z) + d(z))} = \frac{\det \widetilde{\Theta}(z)(\widehat{n}(z) - \widehat{f}(z))}{\widetilde{c}(z)^2(\widehat{n}(z) + \frac{d(z)}{c(z)})(\widehat{f}(z) + \frac{d(z)}{c(z)})} \\ &= \frac{(z - z_1)^{2k}(\widehat{n}(z) - \widehat{f}(z))}{\widetilde{c}(z)^2(\widehat{n}(z) + \frac{d(z)}{c(z)})(\widehat{f}(z) + \frac{d(z)}{c(z)})}, \end{aligned}$$

hence, since $\widetilde{c}(z_1) = \gamma_{k1}$,

$$\frac{\widehat{n}(z) - \widehat{f}(z)}{(\widehat{n}(z) + \frac{d(z)}{c(z)})(\widehat{f}(z) + \frac{d(z)}{c(z)})} = o((z - z_1)^{2(\ell-k)+1}), \quad z \widehat{\rightarrow} z_1.$$

From this and (9.3) we obtain

$$\begin{aligned} \frac{-1}{\widehat{n}(z) + \rho} &= \frac{-1}{\widehat{f}(z) + \rho} + o((z - z_1)^{2(\ell-k)+1}) \left(1 + \frac{\frac{d(z)}{c(z)} - \rho}{\widehat{n}(z) + \rho} \right) \left(1 + \frac{\frac{d(z)}{c(z)} - \rho}{\widehat{f}(z) + \rho} \right) \\ &= \frac{-1}{\widehat{f}(z) + \rho} + o((z - z_1)^{2(\ell-k)+1}), \quad z \widehat{\rightarrow} z_1. \end{aligned} \quad (9.4)$$

The quotient on the left-hand side is a Nevanlinna function and the one on the right-hand side is, like \widehat{f} , a rational Nevanlinna function of degree $\ell - k$. By Theorem 9.1 we now have $\widehat{n} = \widehat{f}$, hence $n = f$.

The remaining combinations $\hat{n} = \infty$, $\hat{f} \in \mathbf{N}_0$ and $\hat{n} \in \mathbf{N}_0$, $\hat{f} = \infty$ do not occur, since

$$\hat{n} = \infty \iff \hat{f} = \infty.$$

This equivalence follows formally from (9.4) and Remark 3.4 (1); it can be shown by going through the above calculations and letting \hat{n} or \hat{f} go to ∞ .

Case (b): $\gamma_{k1} = 0$. Since the parameters \hat{n} and \hat{f} satisfy (5.30), they are bounded Nevanlinna functions, and computations like the ones following (9.3) now show that

$$\hat{n}(z) = \hat{f}(z) + o((z - z_1)^{2(\ell-k)+1}), \quad z \xrightarrow{\widehat{}} z_1.$$

As in 2) of Case (a), it follows again from Theorem 9.1 that $\hat{n} = \hat{f}$, hence $n = f$. □

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